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A NEW METRIC ON SPACE OF FUZZY SETS

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Abstract. In this paper, we introduce a new metric on space $\mathcal{F}(R^p)$ of fuzzy sets and prove that $\mathcal{F}(R^p)$ is separable and complete.

1. Introduction

The Skorokhod metric on the space $D[0,1]$ of functions from $[0,1]$ into the real line R which are right-continuous and have left limits was introduced to study limits theorems for stochastic processes with jumps (see Skorokhod [11]). It turned out that the Skorokhod metric plays a key role for the convergence of probability measures on $D[0,1]$ (see Billingsley [1], Jacod and Shirayaev [6]). Goetschel and Voxman [3] obtained the representation theorem of fuzzy numbers in R which shows similarity between the space $D[0,1]$ and the space $F(R)$ of fuzzy numbers in R. Using this representation theorem, Joo and Kim [7] introduced a metric on $F(R)$ similar to the Skorokhod metric on $D[0,1]$ and proved that $F(R)$ is separable and complete in the metric. Thus it seems natural that we ask whether similar results to those of above mentioned works can also be obtained for the space of fuzzy sets in more general setting.

The purpose of this paper is to answer this question. Section 2 is devoted to describe some basic concepts of fuzzy sets. In section 3, we introduce a new metric on the space $\mathcal{F}(R^p)$ of fuzzy sets in R^p and prove that $\mathcal{F}(R^p)$ is separable and complete in this metric.

2. Preliminaries

In this section, we describe some preliminary results of fuzzy sets.

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Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. Then the space $\mathcal{K}(R^p)$ is metrizable by the Hausdorff metric defined by

$$d_H(A,B) \;=\; \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\}$$

where $|.|$ denotes the Euclidean norm. It is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric d_H (See Debreu [4]).

LEMMA 2.1. *Assume that* $\{A_n\}$ *is a increasing (resp. decreasing) sequence in* $\mathcal{K}(R^p)$. *If there is a subsequence of* $\{A_n\}$ *converging to* $A \in \mathcal{K}(R^p)$ *with respect to the Hausdorff metric* d_H, *then*

$$\lim_{n\to\infty} d_H(A_n, A) \;=\; 0.$$

In addition,

$$A \;=\; \bigcap_{n\geq 1}\overline{\bigcup_{k\geq n} A_k}$$

where $\bar{B}$ *denotes the closure of* B *in* R^p.

Proof. See Lemma 2.2 of Kaleva [8]. □

Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper semicontinuous;
(3) $supp\,\tilde{u} = \overline{\{x \in R^p : \tilde{u}(x) > 0\}}$ is compact.

For a fuzzy set $\tilde{u}$ in R^p, if we define

$$L_\alpha\tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

then, it follows immediately that $\tilde{u} \in \mathcal{F}(R^p)$ if and only if $L_\alpha\tilde{u} \in \mathcal{K}(R^p)$ for each $\alpha \in [0,1]$.

LEMMA 2.2. *For* $\tilde{u} \in \mathcal{F}(R^p)$, *we define*

$$f : [0,1] \longrightarrow (\mathcal{K}(R^p), d_H), f(\alpha) = L_\alpha \tilde{u}.$$

Then the followings hold;

(1) f *is left continuous on* $(0,1]$,
(2) f *has right-limits on* $[0,1)$*and* f *is right-continuous at* 0.

Proof. (1). Let $\alpha \in (0,1]$ and $\{\alpha_n\}$ be an increasing sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. Then $\{L_{\alpha_n}\tilde{u}\}$ is a decreasing sequence in $\mathcal{K}(R^p)$ with $L_{\alpha_n}\tilde{u} \subset L_0\tilde{u}$ for all n. Since $L_0\tilde{u}$ is compact, the sequence $\{L_{\alpha_n}\tilde{u}\}$ has a convergent subsequence in $(\mathcal{K}(R^p), d_H)$. By Lemma 2.1, it converges to the limit

$$\bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} L_{\alpha_k}\tilde{u}} = \bigcap_{n\geq 1} L_{\alpha_n}\tilde{u} = L_\alpha \tilde{u}$$

Thus, f is left-continuous at α.
(2). Let $\alpha \in [0,1)$ and $\{\alpha_n\}$ be a decreasing sequence in $[0,1]$ with $\alpha_n > \alpha$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. By the similar arguments as in (1), we have that $\{L_{\alpha_n}\tilde{u}\}$ converges to the limit

$$\bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} L_{\alpha_k}\tilde{u}} = \overline{\bigcup_{\beta>\alpha} L_\beta \tilde{u}}$$

Hence, f has right-limit at α. The right-continuity of f at 0 follows immediately. □

We denote $\overline{\bigcup_{\beta>\alpha} L_\beta \tilde{u}}$ by $L_{\alpha^+}\tilde{u}$. Now we define, for $J \subset [0,1]$,

$$w_{\tilde{u}}(J) = \sup_{\alpha_1,\alpha_2 \in J} d_H(L_{\alpha_1}\tilde{u}, L_{\alpha_2}\tilde{u}) \tag{2.1}$$

then it follows that for $0 \leq \alpha < \beta \leq 1$,

$$w_{\tilde{u}}(\alpha,\beta) = w_{\tilde{u}}(\alpha,\beta] = d_H(L_{\alpha^+}\tilde{u}, L_\beta \tilde{u}),$$

and

$$w_{\tilde{u}}[\alpha,\beta) = w_{\tilde{u}}[\alpha,\beta] = d_H(L_\alpha\tilde{u}, L_\beta\tilde{u}).$$

Also,

$$\lim_{\beta\to\alpha^+} w_{\tilde{u}}(\alpha,\beta] = 0,$$
$$\lim_{\beta\to\alpha^+} w_{\tilde{u}}[\alpha,\beta] = d_H(L_\alpha\tilde{u}, L_{\alpha^+}\tilde{u}),$$

and

$$\lim_{\gamma\to\alpha^-} w_{\tilde{u}}(\gamma,\alpha] = \lim_{\gamma\to\alpha^-} w_{\tilde{u}}[\gamma,\alpha] = 0.$$

Thus, if we define

$$w_{\tilde{u}}(\alpha) = d_H(L_\alpha\tilde{u}, L_{\alpha^+}\tilde{u}), \tag{2.2}$$

then

$$w_{\tilde{u}}[\alpha,\beta) \le w_{\tilde{u}}(\alpha) + w_{\tilde{u}}(\alpha,\beta),$$

and the function f defined as in Lemma 2.2 is continuous at α if and only if $w_{\tilde{u}}(\alpha) = 0$.

Now, we define the metric d_∞ on $\mathcal{F}(R^p)$ by

$$d_\infty(\tilde{u},\tilde{v}) = \sup_{0\le\alpha\le1} d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) \tag{2.3}$$

Then it is well-known that $\mathcal{F}(R^p)$ is complete but not separable with respect to d_∞(see Klement, Puri and Ralescu [10]).

3. Main Results

In this section, we introduce a new metric on $\mathcal{F}(R^p)$ and prove that $\mathcal{F}(R^p)$ is a Polish space. To this end, we will proceed by similar arguments in Joo and Kim [7]. First, let T denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u},\tilde{v} \in \mathcal{F}(R^p)$, we define

$$d_s(\tilde{u},\tilde{v}) = \inf\{\epsilon : \text{there exists a } t \text{ in } T \text{ such that} \quad \sup_{0\le\alpha\le1} |t(\alpha)-\alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\} \tag{3.1}$$

where d_∞ is the metric on $\mathcal{F}(R^p)$ defined by (2.3). Then it follows immediately that d_s is a metric on $\mathcal{F}(R^p)$ and $d_s(\tilde{u}, \tilde{v})$. The topology generated by the metric d_s will be called the Skorokhod topology. Then it follows that a sequence $\{\tilde{u}_n\}$ in $\mathcal{F}(R^p)$ converges to a limit $\tilde{u}$ in the metric d_s if and only if there exists a sequence of functions $\{t_n\}$ in T such that

$$\lim_{n\to\infty} t_n(\alpha) = \alpha \text{ uniformly in}\, \alpha,$$

and

$$\lim_{n\to\infty} d_\infty(t_n(\tilde{u}_n), \tilde{u}) \;=\; 0.$$

If $d_\infty(\tilde{u}_n, \tilde{u}) \to 0$, then $d_s(\tilde{u}_n, \tilde{u}) \to 0$. But, the converse is not true (For counter-example, see Joo and Kim [7]).

LEMMA 3.1. *For each $\tilde{u} \in \mathcal{F}(R^p)$ and $\epsilon > 0$, there exist a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$ of $[0, 1]$ such that*

$$w_{\tilde{u}}(\alpha_{i-1}, \alpha_i) < \epsilon,\ i = 1, 2..., r. \tag{3.2}$$

Proof. Let τ be the supremum of those α in $[0, 1]$ for which $[0, \alpha]$ can be decomposed into finitely many subintervals satisfying (3.2). Since the function $f(\alpha) = L_\alpha \tilde{u}$ is right-continuous at 0, we have $\tau > 0$. Also, since $f(\alpha) = L_\alpha \tilde{u}$ is left-continuous at τ, $[0, \tau]$ can itself be so decomposed. Now suppose that $\tau < 1$. Then since $f(\alpha) = L_\alpha \tilde{u}$ has right-limit at τ, there exists $\delta \in (0, 1 - \tau)$ such that

$$w_{\tilde{u}}(\tau, \tau + \delta) \;<\; \epsilon$$

which is impossible because $[0, \tau + \delta]$ can also be decomposed into finitely many subintervals satisfying (3.2) in this case. □

For $\tilde{u} \in \mathcal{F}(R^p)$ and $0 < \delta < 1$, we define

$$w'_{\tilde{u}}(\delta) = \inf \max_{1\le i\le r} w_{\tilde{u}}(\alpha_{i-1}, \alpha_i) \tag{3.3}$$

where the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < ... < \alpha_r$ of $[0, 1]$ satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i. Then, Lemma 3.1 is equivalent to the assertion that

$$\lim_{\delta\to 0} w'_{\tilde{u}}(\delta) = 0$$

for each $\tilde{u} \in \mathcal{F}(R^p)$.

THEOREM 3.2. *$(\mathcal{F}(R^p), d_s)$ is separable.*

Proof. Let $\mathcal{F}_0(R^p)$ be the family of $\tilde{v} \in \mathcal{F}(R^p)$ which for some positive integer k, there exist $S_1 \supset S_2 \supset \cdots \supset S_k$ which are finite unions of p-dimensional cubes of the form $\prod_{j=1}^{p}[a_j, b_j]$ with rational points a_j, b_j such that

$$\tilde{v}(x) = \sum_{i=1}^{k-1} \frac{i}{k} I_{S_i \setminus S_{i+1}}(x) \, + \, I_{S_k}(x)$$

where I_A denotes the indicator function of A.

Then $\mathcal{F}_0(R^p)$ is exactly same as the family of $\tilde{v} \in \mathcal{F}(R^p)$ which $L_\alpha \tilde{v}$ is a finite union of p-dimensional cube with rational vertices and identical over $[0, \frac{1}{k}]$ and $(\frac{i-1}{k}, \frac{i}{k}], 2 \le i \le k$, for some k.

Now it is enough to prove that $\mathcal{F}_0(R^p)$ is dense with respect to d_s. Let $\tilde{u} \in \mathcal{F}(R^p)$ and $\epsilon > 0$ be arbitrary fixed. By lemma 3.1, we can take a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$ of $[0, 1]$ satisfying (3.2). And then we choose a positive integer m so that

$$\frac{1}{m} < \epsilon \text{ and } \frac{1}{m} < (\alpha_i - \alpha_{i-1}) \text{ for all } i$$

Also now we take a finite union of p-dimensional cubes of the form $\prod_{j=1}^{p}[a_j, b_j]$ with rational points a_j, b_j such that

$$S_1 \supset S_2 \supset \cdots \supset S_k$$

and

$$L_{\alpha_i}\tilde{u} \subset S_i \subset N(L_{\alpha_i}\tilde{u}, \epsilon)$$

where $N(A, \epsilon) = \{y \in R^p : \inf_{a \in A} |y - a| < \epsilon\}$.

Let us define $m_i = \min\{j | \alpha_i \le \frac{j}{m}\}$ and take $t \in T$ to $\frac{m_i}{m}$ at the points α_i and be linear in between. Then

$$\sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \frac{1}{m} < \epsilon$$

Now, if we define

$$\tilde{v}(x) = \sum_{i=1}^{r-1} \frac{m_i}{m} I_{S_i \setminus S_{i+1}}(x) + I_{S_r}(x),$$

then $\tilde{v} \in \mathcal{F}_0(R^p)$. Since $L_{\alpha_i}\tilde{u} \subset L_\alpha \tilde{u} \subset L_{\alpha_{i-1}^+}\tilde{u} \subset S_i$ for $\alpha_{i-1} < \alpha \leq \alpha_i$, we have

$$\begin{aligned} \sup_{\alpha_{i-1}<\alpha\leq\alpha_i} d_H(L_\alpha\tilde{u}, L_\alpha t^{-1}(\tilde{v})) &= \sup_{\alpha_{i-1}<\alpha\leq\alpha_i} d_H(L_\alpha\tilde{u}, L_{t(\alpha)}\tilde{v}) \\ &= \sup_{\alpha_{i-1}<\alpha\leq\alpha_i} d_H(L_\alpha\tilde{u}, S_i) \ \leq \ \epsilon, \end{aligned}$$

which implies $d_\infty(\tilde{u}, t^{-1}(\tilde{v})) \ \leq \ \epsilon$. Therefore we conclude $d_s(\tilde{u},\tilde{v}) \leq \epsilon$.□

THEOREM 3.3. *$\mathcal{F}(R^p)$ is not complete with respect to the metric d_s.*

Proof. Let us define

$$\tilde{u}_n = \begin{cases} 1, & x = \theta \\ \frac{1}{n}, & x \in [0,1]^p, x \neq \theta \\ 0, & \text{elsewhere,} \end{cases}$$

where θ is the zero vector of R^p. If we denote, for distinct positive integers m, n, $t_{m,n}$ is a member of T such that $t_{m,n}(\frac{1}{n}) = \frac{1}{m}$ and linear in elsewhere, then

$$\sup_{0\leq\alpha\leq1} |t_{m,n}(\alpha) - \alpha| = |\frac{1}{n} - \frac{1}{m}|,$$

and

$$d_\infty(t_{m,n}(\tilde{u}_n), \tilde{u}_m) = 0,$$

which implies $d_s(\tilde{u}_n, \tilde{u}_m) \leq |\frac{1}{n} - \frac{1}{m}|$.
On the other hand, $d_\infty(t(\tilde{u}_n), \tilde{u}_m) = 0$ or 1 for each $t \in T$. Thus, if $d_\infty(t(\tilde{u}_n), \tilde{u}_m) = 0$, then $t(\frac{1}{n}) = \frac{1}{m}$ which implies

$$\sup_{0\leq\alpha\leq1} |t(\alpha) - \alpha| \geq |\frac{1}{n} - \frac{1}{m}|$$

Hence $d_s(\tilde{u}_n, \tilde{u}_m) = |\frac{1}{n} - \frac{1}{m}|$ and $\{\tilde{u}_n\}$ is a Cauchy sequence in the metric d_s.

Now, we prove that $\{\tilde{u}_n\}$ is not convergent in the metric d_s. To this end, it suffices to show that if $t \in T$ and $\tilde{u} \in \mathcal{F}(R^p)$, then there exists $\epsilon_0 > 0$ such that $d_\infty(t(\tilde{u}), \tilde{u}_n) \geq \epsilon_0$ for infinitely many n . Suppose that this assertion is not true. Then

$$\begin{aligned}\sup_{\frac{1}{n}<\alpha\leq 1} \|L_{t^{-1}(\alpha)}\tilde{u}\| &= \sup_{\frac{1}{n}<\alpha\leq 1} d_H(L_{t^{-1}(\alpha)}\tilde{u}, L_\alpha \tilde{u}_n) \\ &\leq d_\infty(t(\tilde{u}), \tilde{u}_n) \to 0 \text{ as } n \to \infty.\end{aligned}$$

This means that

$$\sup_{0<\alpha\leq 1} \|L_{t^{-1}(\alpha)}\tilde{u}\| = 0,$$

that is,

$$L_\alpha \tilde{u} = \{\theta\} \text{ for each } \alpha > 0.$$

Therefore,

$$\tilde{u}(x) = \begin{cases} 1, & \text{if } x = \theta \\ 0, & \text{otherwise,} \end{cases}$$

which yields a contradiction since then $d_\infty(t(\tilde{u}), \tilde{u}_n) = 1$. □

Now we shall introduce another metric d_s^* on $\mathcal{F}(R^p)$ which is equivalent to d_s but $\mathcal{F}(R^p)$ is complete with respect to d_s^*. First, for $t \in T$, put

$$|||t||| = \sup_{\alpha\neq\beta} |\log \frac{t(\beta) - t(\alpha)}{\beta - \alpha}|$$

and for $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$, we define

$$\begin{aligned}d_s^*(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 : &\text{ there exists a } t \in T \text{ with } |||t||| \leq \epsilon \\ &\text{such that } d_\infty(\tilde{u}, t(\tilde{v})) \leq \epsilon\}.\end{aligned} \tag{3.4}$$

Then by a very similar manner in Joo and Kim [7], we can obtain the following results, which are listed without proofs.

Theorem 3.4. *d_s^* is a metric on $\mathcal{F}(R^p)$.*

Lemma 3.5. *If $d_s^*(\tilde{u}, \tilde{v}) < \frac{1}{4}$, then $d_s(\tilde{u}, \tilde{v}) \leq 2d_s^*(\tilde{u}, \tilde{v})$.*

Lemma 3.6. *If $0 < \delta < \frac{1}{4}$ and $d_s(\tilde{u}, \tilde{v}) < \delta^2$, then*

$$d_s^*(\tilde{u}, \tilde{v}) \leq 4\delta + w'_{\tilde{u}}(\delta).$$

Theorem 3.7. *The metrics d_s and ${d_s}^*$ on $\mathcal{F}(R^P)$ are equivalent.*

Theorem 3.8. *$\mathcal{F}(R^P)$ is complete with respect to the metric d_s^*.*

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