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ON THE CURIE-WEISS MODEL WITH A NEW HAMILTONIAN

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Abstract. In this paper we obtain similar limit theorems of the Generalized Curie - Weiss model for a new class Hamiltonian. We expressed the saddlepoint approximation by large deviation rate and then obtain the limit theorems.

1. Introduction

A ferromagnetic crystal consists of a large number of sites. The amount of magnetic spin present at site i will be denoted by $X_i^{(n)}, i = 1, 2, \cdots, n$, where n is a positive integer. The magnetic spin present at any site to some dependency among the random variables $X_i^{(n)}$'s. The joint distribution, at a fixed temperature $T > 0$, of the spin random variables $(X_1^{(n)}, \cdots, X_n^{(n)})$, is given by

$$dQ(x_1, \cdots, x_n) = z_n^{-1} \exp\Big\{ - \frac{H_n(x_1, \cdots, x_n)}{T} \Big\} \prod_{i=1}^{n} dP(x_i), \tag{1.1}$$

where P is a probability measure on $\mathbb{R}$. The function $H_n(x_1, \cdots, x_n)$ is known as the Hamiltonian and it represents the energy of the crystal at the configuration $(x_1, \cdots, x_n)$ and z_n is a normalizing constant which is also known as the partition function. Simon and Griffiths(1973) has a particular model in which the Hamiltonian is taken as $H_n(x_1, \cdots, x_n) = -(x_1 + \cdots + x_n)^2/2n$ and P is replaced by P_T in (1.1) where $P_T(x) = P(x\sqrt{T})$. This is known as the Curie - Weiss

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model. And they obtained the asymptotic distribution of the total magnetism $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$ for this model when P is a symmetric Bernoulli measure, i.e., $P(\{1\}) = P(\{-1\}) = 1/2$. The asymptotic distribution of the partial sum of spin random variables in the Curie-Weiss model was obtained by Simon and Grifiths(1973). Ellis and Newman(1978a, 1978b) extended the result of Simon and Griffiths to a large class of probability measures. Jeon(1979) gave a statistically motivated proof of this result and used the technique of the proof to obtain similar limit theorems for a wider class of Hamiltonians, $H_n(x_1, \cdots, x_n) = -\log \varphi^n(s_n/n)$, where φ is a moment generating function of a random variable Y. He also removed the assumption that P is spherically symmetric. The results of Richter(1957) on sums of independent, identically distributed random variables are generalized to arbitrary sequences of random variables by Chaganty and Sethuraman(1985). Using the local limit theorem of Chaganty and Sethuraman(1985) they obtain similar limit theorems for a wider class of Hamiltonians, which are functions of moment generating functions of suitable random variables. Utilizing the results of Daniels (1954) that is the saddlepoint approximation of independent, identically distributed random variables, Lee, Kim and Joen(1993) obtain the dual limit theorems of the generalized Curie - Weiss model.

In this paper, we obtain the similar limit theorems for the generalized Curie - Weiss model which were studied in Lee, Jeon and Kim(1993). For this, we use the results of Jenson(1995) which generalized saddlepoint method of arbitrary sequence of random variables and obtain similar results for wider class of Hamiltonians.

2. Saddlepoint approximation

For a random variable $X \in \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ the moment generating function $\varphi_P(\theta)$ for $\theta \in \mathbb{R}$ is defined as

$$\varphi_P(\theta) = E \exp\{\theta X\} = \int \exp\{\theta X\} P(d\omega).$$

The domain of the transform is $\Theta_P = \{\theta \in \mathbb{R} : \varphi_P(\theta) < \infty\}$. The exponential family generated by X and P consists of probability measures

P_θ, $\theta \in \Theta_P$, given by

$$\frac{dP_\theta(\omega)}{dP} = \varphi^{-1}(\theta)\exp\{\theta X(\omega)\}.$$

The cumulant generating function(c.g.f.) is $\psi_P(\theta) = \log\varphi_P(\theta)$. More generally, the moment generating function under P_θ is $E_\theta \exp\{\xi X\} = \varphi_P(\xi+\theta)/\varphi_P(\theta)$, $\mu_P(\theta) = E_\theta X$ and $\sigma_P^2(\theta) = Var_\theta(X)$ are the mean and variance-covariance matrix of X under P_θ, respectively. That is,

$$\begin{aligned}\mu_P(\theta) = E_\theta X = \int X dP_\theta = \int X \frac{dP_\theta}{dP} dP \\ = \int X\varphi_P^{-1}(\theta)\exp\{\theta X\}dP = \frac{\partial\psi_P(\theta)}{\partial\theta}\end{aligned}$$

and

$$\sigma_P^2(\theta) = Var_\theta(X) = \frac{\partial^2\psi_P(\theta)}{\partial\theta^2}.$$

Denote $L_P(x)$ the Legendre-Fenchel transform of $\psi_P(\theta)$ by

$$L_P(x) = \sup_{\theta\in\Theta_p} \ell_P(\theta : x) = \sup_{\theta\in\Theta_p}\{\theta x - \psi_P(\theta)\}, \tag{2.1}$$

that is, the large deviation rate of $\psi_P(\theta)$ since $\psi_P(\theta)$ is a cumulant generating function. The exponential family is called *regular* if Θ_P is open, that is, $\Theta_P = \text{int}\Theta_P$. The $\ell_P(\theta : x)$ attains its supremum at a point $\hat\theta(x)$, that is, $L_P(x) = \ell_P(\hat\theta(x); x)$ if and only if $x \in \text{int}C$, where C is the closed convex hull of the support of X. The point $\hat\theta(x)$, $x \in \text{int}C$, is unique and the solution of $\mu_P(\theta) = x$ if $x \in \{\mu_P(\theta) : \theta \in \Theta_P\} \subset \text{int}C$. When the exponential family is regular we have in fact that $\{\mu_P(\theta) : \theta \in \Theta_P\} = \text{int}C$, that is, $\mu_P(\theta) = x$ has a solution $\theta \in \Theta_P$ for any $x \in \text{int}C$. This is the saddlepoint. We consider only those values of x for which there exists $\theta = \hat\theta(x) \in \text{int}\Theta_P$ with $\mu_P(\theta) = x$.

Let $\{\alpha_{n\geq 1} : \alpha_n \in \mathbb{Z}^+\}$ be a sequence of positive integers such that α_n is strictly increasing as $n \to \infty$. Let $Y_{\alpha_n} \in \mathbb{R}$, $\alpha_n \in \mathbb{Z}^+$, be a non-lattice random variable with the moment generating function $\varphi_{Y_{\alpha_n}}(\theta) = E\exp\{\theta Y_{\alpha_n}\}$ defined for

$$\theta \in \Theta_{Y_{\alpha_n}} = \{\theta \in \mathbb{R} : \varphi_{Y_{\alpha_n}}(\theta) < \infty\}.$$

Furthermore, we let $\psi_{Y_{\alpha_n}}(\theta) = \frac{1}{\alpha_n}\log\varphi_{Y_{\alpha_n}}(\theta)$, $\mu_{Y_{\alpha_n}}(\theta) = \frac{1}{\alpha_n}\frac{d}{d\theta}\psi_{Y_{\alpha_n}}(\theta)$ and $\sigma^2_{Y_{\alpha_n}}(\theta) = \frac{1}{\alpha_n}\frac{d^2}{d\theta^2}\psi_{Y_{\alpha_n}}(\theta)$. And let $\{m_{\alpha_n} : \alpha_n \geq 1\}$ be a sequence of real numbers in $\mathbb{R}$ such that m_{α_n} converges to m, as $n \to \infty$, where $m_{\alpha_n} = \psi'_{Y_{\alpha_m}}(\theta_{\alpha_n}), \theta_{\alpha_n} \in \Theta_{Y_{\alpha_n}}$ and $m = \psi'_{Y_{\alpha_n}}(\zeta_{\alpha_n})$ such that $\zeta_{\alpha_n} \in \cap\Theta_{Y_{\alpha_n}}$ for all $\alpha_n \geq 1$. The saddlepoint $\theta = \hat{\theta}(x)$ will be determined by $\mu_{Y_{\alpha_n}}(\theta) = x$. Let $Y_n = Y_{\alpha_n}/\alpha_n$. Then the saddlepoint approximation to the density $f_n(x)$ of $Y_n = Y_{\alpha_n}/\alpha_n$ at x is

$$
\begin{aligned}
f_n(x) &= \frac{\alpha_n^{1/2}}{\sqrt{2\pi}\sigma_{\alpha_n}(\theta)} \\
&\qquad \times \exp\{-\alpha_n(\theta x - \psi_{Y_{\alpha_n}}(\theta))\}[1 + o(1)] \\
&= \frac{\alpha_n^{1/2}}{\sqrt{2\pi}\sigma_{\alpha_n}(\theta)} \\
&\qquad \times \exp\{-\alpha_n L_{Y_{\alpha_n}}(x))\}[1 + o(1)] \qquad \text{as } \alpha_n \to \infty.
\end{aligned} \tag{2.2}
$$

REMARK 2.1. In the i.i.d. cases, $Y_{\alpha_n} = Y'_1 + \cdots + Y'_{\alpha_n}$, we obtain $\varphi_{Y_{\alpha_n}}(\theta) = (\varphi(\theta))^{\alpha_n}$, with $\varphi(\theta) = E\exp\{\theta Y'_1\}$. And also we see that $\Theta_{Y_{\alpha_n}} = \{\theta \in \mathbb{R} : \varphi_{Y_{\alpha_n}}(\theta) < \infty\} = \{\theta \in \mathbb{R} : \varphi(\theta) < \infty\}$, $\mu_{Y_{\alpha_n}}(\theta) = \mu(\theta)$ and $\sigma^2_{Y_{\alpha_n}}(\theta) = \sigma^2(\theta)$.

REMARK 2.2. Under the some conditions there exists a subset Θ_0 of $\Theta_{Y_{\alpha_n}}$ for all α_n, saddlepoint approximation holds uniformly for $\theta \in \Theta_0$ and has relative error $O(\alpha_n^{-2})$. See Jenson(1995).

3. Preliminaries

For a probability measure Q on $\mathbb{R}$ with the moment generating function $\varphi_Q(\theta) = \int_{\mathbb{R}} \exp\{tx\}Q(dx)$ for $\theta \in \Theta_Q$, let $\mathcal{M}_Q$ be a class of probability measure P such that for $\int_{\Theta_Q} \varphi_Q(\theta)P(d\theta) < \infty$, where P is a probability measure on $\mathbb{R}$.

Let $X_i^{(n)}$, $i = 1, \cdots, n$, denote a triangular array of random variables with the joint distribution given by

$$
dQ_n(x_1, \cdots, x_n) = z_n^{-1}\exp\{\psi_Q[(x_1 + \cdots + x_n)/n]\}\prod_{i=1}^{n} dP(x_i), \tag{3.1}
$$

where $x_i \in \mathbb{R},\ i = 1, \cdots, n,\ z_n$ is a normalized constant and $P \in \mathcal{M}_Q$. This model is called *the generalized Curie-Weiss Model* which is a direct generalization of the Curie-Weiss model,

$$(3.2) \quad dQ_n(x_1, \cdots, x_n) = z_n^{-1} \exp\{|x_1 + \cdots + x_n|^2/2n]\} \prod_{i=1}^{n} dP(x_i).$$

We now present the theorem of Sethuraman(1961) which was crucially used to obtain the marginal limiting distribution in the proof of main theorem.

THEOREM 3.1. *(Sethuraman;1961) Let λ_n be a sequence of probability measures on $U \times V$, where U and V are topological spaces. Let μ_n be the marginal probability measure of λ_n on U and $\nu_n(u, \cdot)$ be a conditional probability measure on V. Suppose that μ_n converges to a probability measure μ for every measurable set in U and for almost all u with respect to μ, $\nu_n(u, \cdot)$ converges weakly to $\nu(u, \cdot)$. Then λ_n converges weakly to λ, where $\lambda(A \times B) = \int_A \nu(u, B) d\mu(u)$, for every measurable rectangle set $A \times B$.*

4. Main Results

Let $\{Y_{\alpha_n} : n \geq 1\}$ be a fixed sequence of random variables in $\mathbb{R}$ having a saddlepoint approximation (2.2) in section 2. and $E(Y_{\alpha_n}) = m_{\alpha_n}$ and $Var(Y_{\alpha_n}) = \alpha_n \sigma^2$ for all $\alpha_n \geq 1$. Let Q be a probability measure on $\mathbb{R}$ with moment generating function $\varphi_Q(\theta)$ and cumulant generating function $\psi_Q(\theta)$. For given a random variable Y_{α_n} and a probability measure $P \in \mathcal{M}_Q$, define

$$G_{Y_{\alpha_n}}(u) = L_{Y_{\alpha_n}}(u) - \psi_P(u) \qquad \text{for all} \ \ u \in \Theta_P,$$

where $L_{Y_{\alpha_n}}(u)$ is the large deviation rate of the random variable Y_{α_n} and $\psi_P(\cdot)$ is the cumulant generating function of the probability measure P.

DEFINITION 4.1. A real number m is said to be a global minimum for $G_{Y_{\alpha_n}}(\cdot)$ if

$$G_{Y_{\alpha_n}}(u) \geq G_{Y_{\alpha_n}(m)} \text{ for all } u.$$

DEFINITION 4.2. A global minimum m of $G_{Y_{\alpha_n}}$ is said to be of type k if

$$G_{Y_{\alpha_n}}(u+m) - G_{Y_{\alpha_n}}(m) = G^{(2k)}(u) + o(|u|^{2k}) \text{ as } |u| \to 0,$$

where $G^{(2k)}_{Y_{\alpha_n}} \equiv c_{2k}u^{2k}$ and $c_{2k} \equiv G^{(2k)}_{Y_{\alpha_n}}$ for $u \in \mathbb{R}$.

Let $X_1^{(\alpha_n)}, \cdots, X_{\alpha_n}^{(\alpha_n)}, \alpha_n \in \mathbb{Z}^+$, be an array of random variables with the joint distribution

$$dQ_{\alpha_n}(x_1, \cdots, x_{\alpha_n}) = z_{\alpha_n}^{-1} \exp[\alpha_n \psi_{Y_{\alpha_n}}(s_{\alpha_n}/\alpha_n)] \prod_{j=1}^{\alpha_n} dP(x_j), \tag{4.1}$$

where z_{α_n} is a normalizing constant, and $P \in \mathcal{M}_{Y_{\alpha_n}}$. Here, each $x_j \in \mathbb{R}, j = 1, 2, \cdots$ and $s_{\alpha_n} = x_1 + \cdots + x_{\alpha_n}$. Let $S_{\alpha_n} = X_1^{(\alpha_n)} + \cdots + X_{\alpha_n}^{(\alpha_n)}$.

The model (4.1) is the extended model of Chaganty and Sethuraman(1985) and has the larger than Hamiltonian in the model.

DEFINITION 4.3. Let $\mathcal{M}^*$ be the class of probability measures P defined on $\mathbb{R}$ satisfying the following two conditions

$$\int_{\mathbb{R}} \exp\{\psi_{Y_{\alpha_n}}(u)\} dP(u) < \infty \tag{4.2}$$

and

(4.3)

$$\text{There exists } p, l > 0 \text{ such that } \int_{\mathbb{R}} \exp\{-l G_{Y_{\alpha_n}}(u)\} du = O(\alpha_n^p).$$

Note that Q_{α_n} is well defined because

$$\begin{aligned} z_{\alpha_n} &= \int \cdots \int \exp\{\alpha_n \psi_{Y_{\alpha_n}}(s_{\alpha_n}/\alpha_n)\} \prod_{j=1}^{\alpha_n} dP(x_j) \\ &\le \int \cdots \int \exp\{\sum_{j=1}^{\alpha_n} \psi_{Y_{\alpha_n}}(x_j)\} \prod_{j=1}^{\alpha_n} dP(x_j) \\ &= \prod_{j=1}^{\alpha_n} \exp\{\psi_{Y_{\alpha_n}}(x_j)\} < \infty. \end{aligned}$$

THEOREM 4.4. *Let* $P \in \mathcal{M}^*$ *and* $X_1^{(\alpha_n)}, \cdots, X_{\alpha_n}^{(\alpha_n)}$, $\alpha_n \in \mathbb{Z}^+$ *be an array of random variables with the joint distribution (4.1). Let* $m_{\alpha_n} \to m$ *as* $n \to \infty$*. Assume that the following conditions hold,*

(C1) $G_{Y_{\alpha_n}}$ *has a global minimum of type* k *at* m_{α_n}*, for all* α_n*.*

(C2) $G_{Y_{\alpha_n}}^{(2k)}(m_{\alpha_n}) = c_{2k,\alpha_n} \to c_{2k}$ *as* $n \to \infty$*.*

(C3) *There exists* $\epsilon > 0$ *such that* $G_{Y_{\alpha_n}}^{(2k)}(u) \geq \epsilon |u|^{2k}$*, where* $G^{(2k)}(u) = c_{2k}u^{2k}$ *for* $u \in \mathbb{R}$*. Then,*

(4.4)
$$\frac{S_{\alpha_n} - \alpha_n \theta_{\alpha_n}}{\alpha_n^{1-1/2k}} \xrightarrow{d} \begin{cases} z_k^{-1} \exp[-c_{2k}y^{2k}/[\psi''_P(m)]^{2k}(2k)!] & \text{if } k \geq 2 \\ N(0, \psi''_P(m)[\psi''_P(m) - c_2]/c_2) & \text{if } k = 1 \end{cases}$$

where z_k *is a normalizing constant,* $\psi'_P(m_{\alpha_n}) = \theta_{\alpha_n}$*,* $c_2 = G_{Y_{\alpha_n}}^{(2)}(m)$ *and* $c_{2k} = G_{Y_{\alpha_n}}^{(2k)}(m) > 0$*.*

The proof of this theorem follows from a sequences of following lemmas. Let

$$\begin{aligned} h_n(u) &= (2\pi)^{1/2}\sigma_{Y_{\alpha_n}}(\zeta_{\alpha_n})\alpha_n^{-1/2} \\ &\quad \times \exp\{\alpha_n[\psi_P(m_{\alpha_n} + \alpha_n^{-1/2k}u) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\} \\ &\quad \times f_n(m_{\alpha_n} + \alpha_n^{-1/2k}u) \end{aligned} \tag{4.5}$$

and

$$h(u) = \exp\{-G^{(2k)}(u)\} \qquad \text{for all } u, \tag{4.6}$$

where $\psi'_P(\zeta_{\alpha_n}) = m$, $\zeta_{\alpha_n} \in \Theta_{Y_{\alpha_n}}$ for all $\alpha_n \geq 1$.

LEMMA 4.5. *Let* $G_{Y_{\alpha_n}}^{(2k)}(m_{\alpha_n}) = c_{2k,\alpha_n}$*. If* $G_{Y_{\alpha_n}}$ *has a global minimum of type* k *at* m_{α_n} *and* $c_{2k,\alpha_n} \to c_{2k}$*. Then* $h_n(u) \to h(u)$*, as* $n \to \infty$*.*

Proof. Fix $u \in \mathbb{R}$. By the saddlepoint approximation (2.2) we have as $n \to \infty$,

$$\begin{aligned} h_n(u) &= (2\pi)^{1/2}\sigma_{Y_{\alpha_n}}(\zeta_{\alpha_n})\alpha_n^{-1/2} \\ &\qquad \times \exp\{\alpha_n[\psi_P(m_{\alpha_n}+\alpha_n^{-1/2k}u) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\} \\ &\qquad \times \alpha_n^{1/2}(2\pi)^{-1/2}\sigma^{-1}_{Y_{\alpha_n}}(\zeta_{\alpha_n}) \\ &\qquad \times \exp\{-\alpha_n L_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u)\}[1+o(1)] \\ &= \exp\{-\alpha_n[G_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\}[1+o(1)] \\ &= \exp\{-\alpha_n[\alpha_n^{-1}G^{(2k)}(\alpha_n^{-1/2k}u) + o(\alpha_n^{-1}|u|^{2k})]\}[1+o(1)] \\ &= \exp\{-G^{(2k)}(u)\}[1+o(1)]. \end{aligned}$$

The lemma is now immediate taking limits as $n \to \infty$. □

LEMMA 4.6. *Let $0 < \delta < 1/2k$. Suppose that $G_{Y_{\alpha_n}}$'s have a unique global minimum of type k at a point m_{α_n} then there exists α_{N_0} such that for all $\alpha_n \le \alpha_{N_0}$,*

$$\begin{aligned} \alpha_n[G_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u) - G_{Y_{\alpha_n}}(m_{\alpha_n})] \\ \ge G^{(2k)}(u)/2 \text{ uniformly for } |u| < \alpha_n^{\delta}. \end{aligned}$$

Proof. By condition (C2), we can find α_{N_1} such that for $\alpha_n \ge \alpha_{N_1}$,

$$|G^{(2k)}_{Y_{\alpha_n}}(u) - G^{(2k)}(u)| > \epsilon|u|^{2k}/4 \quad \text{for all } u \in \mathbb{R}.$$

Hence by condition (C3) there exists α_{N_1} such that

$$G^{(2k)}_{Y_{\alpha_n}}(u) > G^{(2k)}(u)/2 + \epsilon|u|^{2k}/4 \quad \text{for all } \alpha_n \ge \alpha_{N_1}. \tag{4.7}$$

Since $\alpha_n^{-1}|u|^{2k}$ goes to 0 uniformly for $|u| \le \alpha_n^{\delta}$ and $G_{Y_{\alpha_n}}$ has a global minimum at the point m_{α_n} we can find α_{N_2} such that for all $\alpha_n \ge \alpha_{N_2}$,

$$\alpha_n[G_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u) - G_{Y_{\alpha_n}}(m_{\alpha_n})] > G^{(2k)}_{Y_{\alpha_n}}(u) - \epsilon|u|^{2k}/4. \tag{4.8}$$

The lemma follows form (4.7) and (4.8) choosing $N_0 = \max\{N_1, N_2\}$.□

LEMMA 4.7. *Let $0 < \delta < 1/2k$ be fixed. Then*

$$\int_{|u|\le\alpha_n^\delta} h_n(u)du \to \int h(u)du \quad \text{as } n\to\infty. \tag{4.9}$$

Proof. Note that condition (C3) implies $\int h(u)du < \infty$. The proof is completed using the dominated convergence theorem, Lemma 4.5 and Lemma 4.6. □

LEMMA 4.8. *Let $\{Y_{\alpha_n} : \alpha_n \ge 1\}$ be a sequence of random variables satisfying the condition of section 2. Then for all $u \in \mathbb{R}$,*

$$\exp\{\alpha_n L_{Y_{\alpha_n}}(m_{\alpha_n}+u)\} f_n(m_{\alpha_n}+u) = O(\alpha_n^{1/2}), \quad \text{as } n\to\infty. \tag{4.10}$$

Proof. By the definition, as $n\to\infty$,

$$\begin{aligned}&|\exp\{\alpha_n L_{Y_{\alpha_n}}(m_{\alpha_n}+u)\} f_n(m_{\alpha_n}+u)|\\ &\qquad = \Big|\frac{\alpha_n^{1/2}}{(2\pi)^{1/2}\sigma_{Y_{\alpha_n}}(\theta)}(1+o(1))\Big|.\end{aligned}$$ □

LEMMA 4.9. *Let $0 < \delta < 1/2k$ be fixed and $h_n(u)$ be as defined by (4.5). Then*

$$\int_{|u|\ge\alpha_n^\delta} h_n(u)du \to 0 \quad \text{as } n\to\infty. \tag{4.11}$$

Proof. ¿From Lemma 4.8 and condition (C1) we obtain the result. By (4.5) we have

$$\begin{aligned}&\int_{|u|>\alpha_n^\delta} h_n(u)du\\ &= (2\pi)^{1/2}\sigma_{Y_{\alpha_n}}(\zeta_{\alpha_n})\alpha_n^{-1/2}\\ &\quad\times\int_{|u|>\alpha_n^\delta}\exp\{\alpha_n[\psi_P(m_{\alpha_n}+\alpha_n^{-1/2k}u)+G_{Y_{\alpha_n}}(m_{\alpha_n})]\}\\ &\quad\times f_n(m_{\alpha_n}+\alpha_n^{-1/2k}u)du\\ &= (2\pi)^{1/2}\sigma_{Y_{\alpha_n}}(\zeta_{\alpha_n})\alpha_n^{-1/2}\\ &\quad\times\int_{|u|>\alpha_n^\delta}\exp\{-\alpha_n[G_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u)+G_{Y_{\alpha_n}}(m_{\alpha_n})]\\ &\quad+\alpha_n L_{Y_{\alpha_n}}(m_{\alpha_n}+\alpha_n^{-1/2k}u)\}f_n(m_{\alpha_n}+\alpha_n^{-1/2k}u)du\end{aligned}$$

substituting $u' = \alpha_n^{-1/2k} u$, we get

$$\begin{aligned}
&\Big| \int_{|u|>\alpha_n^{\delta}} h_n(u)du \Big| \\
&\le (2\pi)^{1/2} \sigma_{Y_{\alpha_n}}(\zeta_{\alpha_n}) \alpha_n^{-(1-1/k)/2} \\
&\qquad \times \int_{|u|>\alpha_n^{\delta-1/2k}} \big| \exp\{-\alpha_n [G_{Y_{\alpha_n}}(m_{\alpha_n}+u) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\} \big| \\
&\qquad \times |\exp\{\alpha_n L_{Y_{\alpha_n}}(m_{\alpha_n}+u) f_n(m_{\alpha_n}+u)| du \\
&\le O(\alpha_n^{p+(1+1/k)/2}) \\
&\qquad \times \max_{|u|\ge \alpha_n^{\delta-1/2k}} \exp\{-(\alpha_n - l)[G_{Y_{\alpha_n}}(m_{\alpha_n}+u) - G_{Y_{\alpha_n}}(m_{\alpha_n})] \\
&\qquad \times \int \exp\{-(\alpha_n - l)[G_{Y_{\alpha_n}}(m_{\alpha_n}+u) - G_{Y_{\alpha_n}}(m_{\alpha_n})] du \\
&= O(\alpha_n^{p+(1+1/k)/2}) \\
&\qquad \times \max_{|u|\ge \alpha_n^{\delta-1/2k}} \exp\{-(\alpha_n - l)[G_{Y_{\alpha_n}}(m_{\alpha_n}+u) - G_{Y_{\alpha_n}}(m_{\alpha_n})].
\end{aligned}$$

The last inequality follows from Lemma 4.8. Thus we get from condition (4.3)

$$\begin{aligned}
\Big| \int_{|u|>\alpha_n^{\delta}} h_n(u)du \Big| &\le O(\alpha_n^{2p+(1+1/k)/2}) \\
&\quad \times \max_{|u|\ge \alpha_n^{\delta-1/2k}} \exp\{-(\alpha_n - l)[G_{Y_{\alpha_n}}(m_{\alpha_n}+y) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\} \\
&= O(\alpha_n^{2p+(1+1/k)/2}) \exp\{-(\alpha_n - l) R_n\}
\end{aligned}$$

and

$$\begin{aligned}
R_n &= \min_{|u|>\alpha_n^{\delta-1/2k}} [G_{Y_{\alpha_n}}(m_{\alpha_n}+y) - G_{Y_{\alpha_n}}(m_{\alpha_n})] \\
&= \min\{[G_{Y_{\alpha_n}}(\alpha_n^{\delta-1/2k} + m_{\alpha_n}) - G_{Y_{\alpha_n}}(m_{\alpha_n})], \\
&\qquad [G_{Y_{\alpha_n}}(-\alpha_n^{\delta-1/2k} + m_{\alpha_n}) - G_{Y_{\alpha_n}}(m_{\alpha_n})]\} \\
&= \frac{c_{2k,\alpha_n}}{(2k)!} \alpha_n^{2k(\delta-1/2k)} + o(\alpha_n^{2k(\delta-1/2k)}),
\end{aligned}$$

since m_{α_n} is the unique global minimum of $G_{Y_{\alpha_n}}$ and $c_{2k,\alpha_n} > 0$. Hence

$$\Big| \int_{|u|>\alpha_n^{\delta}} h_n(u)du \Big| \le O(\alpha_n^{2p+(1+1/k)/2}) \\ \times \exp\Big\{ -(\alpha_n - l)\Big[\frac{c_{2k,\alpha_n}}{(2k)!}\alpha_n^{2k(\delta-1/2k)} + o(\alpha_n^{2k(\delta-1/2k)})\Big]\Big\}$$

which goes to 0 since $0 < \delta < 1/2k$ and the proof is complete. □

Now we proof the Theorem 4.4. We can express the joint distribution Q_{α_n} as follows;

$$dQ_{\alpha_n}(x_{\alpha_1}, \cdots, x_{\alpha_n}) = z_{\alpha_n}^{-1} \exp[\alpha_n \psi_{Y_{\alpha_n}}(s_{\alpha_n}/\alpha_n)] \prod_{j=1}^{\alpha_n} dP(x_j) \\ = z_{\alpha_n}^{-1} \int_{\mathbb{R}} \exp\{s_{\alpha_n} y\} f_n(y) \prod_{j=1}^{\alpha_n} dP(x_j).$$

Substituting $y = m_{\alpha_n} + \alpha_n^{-1/2k} u$,

$$dQ_{\alpha_n}(x_1, \cdots, x_{\alpha_n}) = z_{\alpha_n}^{-1} \alpha_n^{-1/2k} \int_{\mathbb{R}} \exp\{s_{\alpha_n}(m_{\alpha_n} + \alpha_n^{-1/2k} u)\} \\ \times f_n(m_{\alpha_n} + \alpha_n^{-1/2k} u) \prod_{j=1}^{\alpha_n} dP(x_j) \\ = z_{\alpha_n}^{-1} \alpha_n^{-1/2k} \int_{\mathbb{R}} \exp\{s_{\alpha_n}(m_{\alpha_n} + \alpha_n^{-1/2k} u) - \alpha_n \psi_P(m_{\alpha_n} + \alpha_n^{-1/2k} u)\} \\ \times \exp\{\alpha_n \psi_P(m_{\alpha_n} + \alpha_n^{-1/2k} u)\} f_n(m_{\alpha_n} + \alpha_n^{-1/2k} u) \prod_{j=1}^{\alpha_n} dP(x_j) \\ = \int_{\mathbb{R}} \prod_{j=1}^{\alpha_n} dM_{\alpha_n,u}(x_j) g_n(u) du,$$

where

$$M_{\alpha_n,u}(x_j) = \exp\{(m_{\alpha_n} + \alpha_n^{-1/2k} u))x_j - \psi_P(m_{\alpha_n} + \alpha_n^{-1/2k} u) dP(x_j)$$

and

$$g_n(u) = z_{\alpha_n}^{-1}\alpha_n^{-1/2k}\exp\{\alpha_n\psi_P(m_{\alpha_n}+\alpha_n^{-1/2k}u)\}f_n(m_{\alpha_n}+\alpha_n^{-1/2k}u),$$

where $\psi'_P(\xi_{\alpha_n}) = m_{\alpha_n} + u\alpha_n^{-1/2k}$.

Since $\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dQ_{\alpha_n}(x_1,\cdots,x_{\alpha_n}) = 1$ and $\int_{\mathbb{R}} M_{\alpha_n,u}(x_{\alpha_j}dx_{\alpha_j} = 1$ for each α_j, we see that $\int_{\mathbb{R}} g_n(u)du = 1$. Thus $g_n(u)$ is a density function for each n. For fixed $t\in\mathbb{R}$, the c.g.f. of $(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}$ under $dM_{\alpha_n,u}$ obtained as follows;

$$\begin{aligned}
&\log E_{M_{\alpha_n,u}}\exp\{t(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}\}\\
&=\log E_{M_{\alpha_n,u}}\prod_{j=1}^{\alpha_n}\exp\{t(X_j-\alpha_n\theta_{\alpha_n})\alpha_n^{-(1-1/2k)}\\
&=\alpha_n\log[E_{M_{\alpha_n,u}}\exp\{t(X_j-\alpha_n\theta_{\alpha_n})\alpha_n^{-(1-1/2k)}\}\\
&\quad\times\exp\{(m_{\alpha_n}+\alpha_n^{-1/2k}u)X_j-\psi_P(m_{\alpha_n}+\alpha_n^{-1/2k}u)\}]\\
&=u\psi''_P(m_{\alpha_n})t+t^2\psi''_P(m_{\alpha_n})\alpha_n^{-(1-1/2k)}+o(1).
\end{aligned}$$

since $\psi'_P(m_{\alpha_n}) = \theta_{\alpha_n}$. Taking limits as $n\to\infty$ and noting that $m_{\alpha_n}\to m$, we get

$$\begin{aligned}
&\log E_{M_{\alpha_n,u}}\exp\{u(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}\}\\
&\qquad\longrightarrow\begin{cases}\psi''_P(m)ut & \text{if } k\ge 2\\ \psi''_P(m)ut+\psi''_P(m)t^2/2 & \text{if } k=1.\end{cases}
\end{aligned}$$

This shows that under $M_{\alpha_n,u}$ the limiting distribution of $(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}$ is degenerate at $\psi''_P(m)u$ if $k\le 2$ and $N(\psi''_P(m)u,\psi''_P(m))$ if $k=1$. By Lemmas 4.5, 4.7 and 4.9 we get

$$g_n(u) = \frac{h_n(u)}{\int h_n(u)du}\longrightarrow g(u) = \frac{h(u)}{\int h(u)du}\quad\text{as } n\to\infty,$$

where h_n, h are defined by (4.5) and (4.6). The proof of the Theorem 4.4 is now completed using theorem of Sethuraman(1961) as follows; When $k\le 2$, the limiting distribution of $(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}$ is $\psi''_P(m)U_k$, where $U_k\sim g(u)$ and when $k=1$ we note that $g(u)$ is $N(0,c_{2k}^{-1})$ and thus the limiting distribution of $(S_{\alpha_n}-\alpha_n\theta_{\alpha_n})/\alpha_n^{1-1/2k}$ is $N(0,\psi''_P(m)[\psi''_P(m)-c_2]/c_2)$.

REMARK 4.10. If $\alpha_n = n$ then the result is just the same with theorem of Chaganty and Sethuraman(1987).

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