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TOTAL CURVATURE FOR SOME MINIMAL SURFACES

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Abstract. In this paper, we estimate the total curvature of nonparametric minimal surfaces by using the properties of univalent harmonic mappings defined on $\triangle = \{z : |z| > 1\}$.

1. Introduction

Hengartner and Schober [3] studied the class Σ of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\triangle = \{z : |z| > 1\}$ that are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z|, \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in $\triangle$ and $0 \leq |\beta| < |\alpha|$. In addition, $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$.

Now let Ω be a doubly connected domain in the extended w-plane having the point w $= \infty$ as one of its boundary continua. Let S be a nonparametric surface over Ω given by

$$S = \{(u, v, \varphi(u, v)) : u + iv \in \Omega\}.$$

We shall associate with every solution $\varphi(u, v)$ of the classical equation of minimal surfaces,

$$(1 + \varphi_v^2)\varphi_{uu} - 2\varphi_u\varphi_v\varphi_{uv} + (1 + \varphi_u^2)\varphi_{vv} = 0 \tag{1.2}$$

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and the functions

$$\psi(u,v) = \int \frac{\varphi_u dv - \varphi_v du}{\sqrt{1+\varphi_u^2+\varphi_v^2}};$$

$$F = \varphi + i\psi;$$

$$\omega = \frac{\varphi_u - i\varphi_v}{1+\sqrt{1+\varphi_u^2+\varphi_v^2}}.$$

Note that the integral defining ψ is path-independent by virtue of (1.2) and that ψ is determined but for an additive constant. Also, we have $|\omega| < 1$. The functions F and ω are to be considered as defined on S. While ω is single-valued on this surface, F may be multi-valued, but its branches differ only by constants.

If $\varphi(u,v)$ is interpreted as the potential of a flow of a hypothetical "Chaplygin gas" whose density ρ and speed q are connected by the relation

$$\rho^2(1+q^2) = 1,$$

then ψ is the stream-function, F the complex potential, and $\varphi_u - i\varphi_v$ the conjugate complex velocity (cf.[1]).

From now on, assume that φ is nonconstant; that is, S is not a horizontal plane. Then S is a minimal surface if and only if S admits a conformal reparametrization of the form

$$S = \{(\mathrm{Re}G_1(z), \mathrm{Re}G_2(z), \mathrm{Re}F(z)) : z = x+iy \in \triangle\}, \tag{1.3}$$

where

$$G_1(z) = \frac{1}{2}\int F'\left(\frac{1}{\omega} - \omega\right) dz, \quad G_2(z) = \frac{-i}{2}\int F'\left(\frac{1}{\omega} + \omega\right) dz.$$

The function

$$f(z) = \mathrm{Re}G_1(z) + i\mathrm{Re}G_2(z) = \frac{1}{2}\int \frac{F'(z)}{\omega(z)} dz - \frac{1}{2}\overline{\int \omega(z)F'(z)dz}$$

is a univalent harmonic mapping from $\triangle$ onto Ω with $f(\infty) = \infty$ and $\overline{f_{\overline{z}}}/f_z = -\omega^2$. In $\triangle$, the variables F and ω considered as functions of z are regular analytic, dF/dz and ω are single-valued, and

$(1/\omega)(dF/dz) \neq 0, \infty$. The function $\omega(z)$ is regular at $z = \infty$ and $|\omega(\infty)| < 1$. Furthermore, $F'(z)/\omega(z)$ is regular and different from zero at ∞. Also observe that we may assume f is orientation-preserving and that we may obtain any other set of isothermal parameters by applying a conformal mapping to $\triangle$ (cf.[1]). Since $f(z) \in \Sigma$, f has the representation (1.1).

Note that $\omega(u,v) = 0$ if and only if the unit normal vector to the surface

$$\vec{n} = (-\varphi_u, -\varphi_v, 1)/\sqrt{1+\varphi_u^2+\varphi_v^2}$$

is $(0,0,1)$ at $(u, v, \varphi(u,v))$. This means that S has a horizontal tangent plane at points where ω vanishes.

In this paper, we estimate the total curvature of nonparametric minimal surfaces over Ω by using the properties of univalent harmonic mappings f in $\triangle$ with $f(\infty) = \infty$.

2. Curvature

Let S be a nonparametric minimal surface over Ω, then S is given as in (1.3). The first fundamental form for the Euclidean length on S is

$$ds^2 = \lambda^2 |dz|^2 \qquad \text{where} \quad \lambda = |h' + \frac{A}{2z}| + |g' + \frac{\overline{A}}{2z}|.$$

Therefore the Gaussian curvature K of S is

$$K = \frac{-\Delta \log \lambda}{\lambda^2} = \frac{-|(g'+\frac{\overline{A}}{2z})'(h'+\frac{A}{2z}) - (g'+\frac{\overline{A}}{2z})(h'+\frac{A}{2z})'|^2}{|(h'+\frac{A}{2z})(g'+\frac{\overline{A}}{2z})|\left(|h'+\frac{A}{2z}|+|g'+\frac{\overline{A}}{2z}|\right)^4}.$$

Since $a = \overline{f_{\bar{z}}}/f_z = (g'+\frac{\overline{A}}{2z})/(h'+\frac{A}{2z}) = -\omega^2$, we obtain

$$K = \frac{-4|\omega'|^2}{\left(|h'+\frac{A}{2z}|+|g'+\frac{\overline{A}}{2z}|\right)^2 (1+|\omega|^2)^2}.$$

If we now let D be a domain whose closure is in $\triangle$, the surface over $f[D] (\subset \Omega)$ has total curvature given by

$$T = \iint_D K dA = \iint_D K\lambda^2 dx dy \qquad \text{where} \quad z = x + iy. \tag{2.1}$$

By substituting λ^2 and K into (2.1), we have that

$$T = -\iint_D \frac{4|\omega'|^2}{(|\omega|^2+1)^2} dxdy.$$

The estimate $|\omega'|/(1-|\omega|^2) \leq 1/(|z|^2-1)$ from Schwarz's lemma for $|z|>1$ implies

$$|T| = 4\iint_D \frac{|\omega'|^2}{(|\omega|^2+1)^2} dxdy \leq 4\iint_D \left(\frac{1-|\omega|^2}{1+|\omega|^2}\right)^2 \frac{1}{(|z|^2-1)^2} dxdy.$$

Since $\left(\frac{1-|\omega|^2}{1+|\omega|^2}\right)^2 \leq 1$ for $|\omega|<1$, we obtain

$$|T| \leq \iint_D \frac{4}{(|z|^2-1)^2} dxdy.$$

Consider the surface

$$S_{r_1}^{r_2} = \{(\mathrm{Re}G_1(z), \mathrm{Re}G_2(z), \mathrm{Re}F(z)) : z = x+iy \in D_{r_1}^{r_2}\}$$

where $D_{r_1}^{r_2} = \{z : 1 < r_1 \leq |z| \leq r_2\}$. Then the estimate of $|T|$ for $S_{r_1}^{r_2}$ is given by

$$|T| \leq \frac{4\pi(r_2^2 - r_1^2)}{(r_1^2-1)(r_2^2-1)}.$$

We have proved the following theorem.

Theorem. *Let S be a nonparametric minimal surface over a doubly connected domain Ω. Then an estimate for the total curvature T of $S_{r_1}^{r_2}$ is*

$$|T| \leq \frac{4\pi(r_2^2 - r_1^2)}{(r_1^2-1)(r_2^2-1)}.$$

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