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# ORTHOGONAL GROUPS OF QUATERNION ALGEBRAS

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Abstract. The structure of orthogonal groups of quaternion algebras is studied.

Let $L$ be any field, and $a,\ b \in L^*$. The quaternion algebra $B = \left(\frac{a,b}{L}\right)$ is the $L$-algebra on two generators $i,\ j$ with the defining relations:

$$i^2 = a, \quad j^2 = b, \quad k = ij = -ji.$$

Then $B$ is a central $L$-simple algebra of dimension 4 over $L$ with basis $\{1, i, j, k\}$. Note that if $L'$ is any extension field of $L$, then

$$\left(\frac{a,b}{L}\right) \otimes L' = \left(\frac{a,b}{L'}\right). \tag{1}$$

For any quaternion $x = a_1 + a_2 i + a_3 j + a_4 k$, the conjugate of $x$ is defined by

$$x' = a_1 - a_2 i - a_3 j - a_4 k,$$

and, its reduced norm $N$ and reduced trace $\mathrm{Tr}$ are defined by

$$Nx = xx', \qquad \mathrm{Tr}\, x = x + x'.$$

If we define

$$(x, y)_B = \mathrm{Tr}\,(xy')$$

then $B, (\ ,\ )_B$ becomes a regular quadratic space with an orthogonal basis $\{1, i, j, k\}$ and its matrix is given by $\mathrm{diag}\,(2, -2a, -2b, 2ab)$. Therefore $\det B = 1$ in $L^*/L^{*2}$.

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Now we study the structure of $O(B)$. First we recall the theorem of Cartan-Dieudonné. Let $U, (\ ,\ )$ be any quadratic space. For any anisotropic $u \in U$, the *symmetry* $\tau_u$ is defined by

$$\tau_u(x) = x - \frac{2(x,u)}{(u,u)}u.$$

It is clear that $\det \tau_u = -1$, and $\tau_u^2 = 1$.

THEOREM 1 [CARTAN-DIEUDONNÉ]. *Let $U, (\ ,\ )$ be a regular quadratic space of dimension $n$. Then every isometry in $O(V)$ is a product of at most $n$ symmetries.*

Let $B^*$ be the set of units in $B$. $B^*$ is exactly the set of anisotropic vectors in $B$. For any $u \in B^*$,

$$\tau_u(x) = -ux'(u')^{-1}. \tag{3}$$

Recall that the spinor norm is

$$\theta(\tau_u) = N(u)$$

by definition. Hence it is not difficult to see that rotations, being products of 4 symmetries, have the form

$$\rho(u,v)\colon x \mapsto uxv^{-1}, \tag{4}$$

and reflections, being products of 3 symmetries, have the form

$$\tau(u,v)\colon x \mapsto -ux'v^{-1}, \tag{5}$$

and that

$$\theta(\rho(u,v)) = \theta(\tau(u,v)) = N(u) = N(v), \tag{6}$$

where $u, v \in B^*$ with $N(u) = N(v)$. Next, observe that $\rho(u_1, v_1) \neq \tau(u_2, v_2)$ for any $u_i, v_i \in B^*$, $i = 1, 2$. That is,

$$\rho(u,v) \in SO(B). \tag{7}$$

Otherwise there would be $u, v \in B^*$ such that $-ux'v^{-1} = x$ for all $x \in B$. Taking $x \in L$, we must have $v = -u$. Hence $ux'u^{-1} = x$ for all $x$. Then take $x = u$, so that we have $u = u'$, that is, $u \in L^*$. But then $x' = x$ for all $x$, which is absurd. Let

$$(B^*)_0^2 = \{\, (u,v) \in B^* \times B^* \mid N(u) = N(v) \,\}$$
$$B^1 = \{\, u \in B^* \mid N(u) = 1 \,\}.$$

Theorem 2. *We have an exact sequence*

$$1 \longrightarrow L^* \longrightarrow (B^*)^2_0 \xrightarrow{\ \rho\ } SO(B) \longrightarrow 1, \tag{9}$$

*where $L^*$ is embedded into $(B^*)^2_0$ diagonally. In particular,*

$$SO(B) \simeq (B^*)^2_0/L^*.$$

*Furthermore, the following sequence is also exact:*

$$1 \longrightarrow \rho((B^1)^2) \longrightarrow SO(B) \xrightarrow{\ \theta\ } L^*/L^{*2} \longrightarrow 1. \tag{10}$$

*Proof.* First part follows from the discussion above and the fact that $B$ is central. For the second part we show that $\ker\theta \subset \rho((B^1)^2)$. Let $h = \rho(u, v) \in SO(B)$ with $\theta(h) = N(u) \in L^{*2}$. Write $\alpha^2 = N(u)$ for $\alpha \in L^*$ and let $u_1 = \alpha^{-1}u$, $v_1 = \alpha^{-1}v$. Then $(u_1, v_1) \in (B^1)^2$ and hence $h = \rho(u, v) = \rho(u_1, v_1)$. Now everything else is clear from the discussion above. □

Finally, we note that $O(B)$ is generated by $SO(B)$ and the quaternion conjugation.

## References


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