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TOEPLITZ OPERATORS ON HARMONIC BERGMAN FUNCTIONS ON HALF-SPACES

HeungSu Yi

Abstract. We study Toeplitz operators on the harmonic Bergman Space $b^p(\mathbf{H})$, where $\mathbf{H}$ is the upper half space in $\mathbf{R}^n (n \geq 2)$, for $1 < p < \infty$. We give characterizations for the Toeplitz operators with positive symbols to be bounded.

1. Introduction

The upper half-space $H = H_n$ is the open subset of $\mathbf{R}^n (n \geq 2)$ given by

$$H = \{z = (z', z_n) \in \mathbf{R}^n : \; z_n > 0\},$$

where we have written a typical point $z \in \mathbf{R}^n$ as $z = (z', z_n)$, with $z' \in \mathbf{R}^{n-1}$ and $z_n \in \mathbf{R}$. We fix $z_0 = (0, 1)$ throughout this paper.

For $1 \leq p < \infty$, the harmonic Bergman space $b^p = b^p(\mathbf{H})$ is the space of all complex-valued functions u on $\mathbf{H}$ such that

$$\|u\|_p = \left(\int_{\mathbf{H}} |u|^p \, dV \right)^{1/p} < \infty,$$

where V denotes the Lebesgue volume measure on $\mathbf{H}$. The space b^p is a closed subspace of $L^p(\mathbf{H}, dV)$ and hence a Banach space. In particular b^2 is a Hilbert space and so there is an orthogonal projection Q from the Hilbert space $L^2(\mathbf{H}, dV)$ onto b^2.

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For each $z \in \mathbf{H}$, the map $u \mapsto u(z)$ is a bounded linear functional on b^2. Thus there exists a unique function $R_z(\cdot) = R(z, \cdot) \in b^2$ such that

$$u(z) = \int_{\mathbf{H}} u(w) R(z,w)\, dw$$

for every $u \in b^2$. Here and elsewhere, we use the notation $dw = dV(w)$ for simplicity. The function R on $\mathbf{H} \times \mathbf{H}$ is called the harmonic Bergman kernel of b^2. For $f \in L^2(\mathbf{H}, dV)$ and $z \in \mathbf{H}$, we have

$$Qf(z) = \int_{\mathbf{H}} f(w) R(z,w)\, dw.$$

If $1 < p < \infty$, then Q is a bounded projection from $L^p(\mathbf{H}, dV)$ onto b^p(See [8]). The integral operator Q extends to a class of measures as follows: If μ is a positive Borel measure on $\mathbf{H}$ such that $(1 + |z|^n)^{-1} \in L^1(\mu)$, then $R_z \in L^1(\mu)$ for all $z \in \mathbf{H}$ so that the integral

$$Q\mu(z) = \int_{\mathbf{H}} R(z,w)\, d\mu(w)$$

defines a harmonic function on $\mathbf{H}$. Let M^+ be the set of all positive Borel measures such that

$$\int_{\mathbf{H}} \frac{d\mu(z)}{1 + |z|^n} < \infty.$$

For $\mu \in M^+$, we define an operator T_μ by

$$T_\mu u = Q(u d\mu)$$

for $u \in b^p$ and for $1 < p < \infty$. This operator T_μ is called the *Toeplitz operator* with *symbol* μ. If $d\mu = f dV$, we simply write $T_f = T_\mu$. Even though T_μ may not be defined on all of b^p, it is always densely defined on b^p.

Toeplitz operators on analytic Bergman spaces have been well studied; See [2], [6] and [9]. In a recent paper, Miao [7] obtained the analogous results for positive Toeplitz operators on the harmonic Bergman spaces of the ball. In this paper we consider positive Toeplitz operators on b^p for $1 < p < \infty$ on $\mathbf{H}$ to be bounded. This paper is written as follows. In section 2, we study the harmonic Bergman kernel $R(z, w)$ and we prove some basic lemmas which will be used in the next section. In section 3, we prove the main result of this paper, the characterizations of boundedness of positive Toeplitz operators.

Throughout this paper we will use the same constant $C > 0$ to denotes various constants, often with subscripts indicating dependency, which may vary at each occurrence. When $1 < p < \infty$, we use p' to denote the index conjugate of p, i.e., $1/p + 1/p' = 1$. We use the symbol $\approx$ to indicate that the quotient of two quantities is bounded above and below by inessential constants when the variable varies.

2. Preliminaries and Basic Lemmas

In this section, we review some preliminaries from [3], [4] and [8]. We also prove in this section some lemmas which we will use later. The explicit formula for harmonic Bergman kernel $R(z,w)$ is given in [3] by

$$R(z,w) = \frac{4}{n\sigma_n} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}$$

for $z, w \in \mathbf{H}$, where σ_n denotes the volume of the unit ball of $\mathbf{R}^n$ and $\overline{w} = (w', -w_n)$. From this formula, we can see easily that

$$|R(z,w)| \leq \frac{C_n}{|z - \overline{w}|^n} \tag{2.1}$$

for all $z, w \in \mathbf{H}$.

Let $\mathcal{R} = \text{span}\{R(z,\cdot) : z \in \mathbf{H}\}$ and let D_0 denote the space of all harmonic functions u on $\mathbf{H}$ such that

$$\sup_{z \in \mathbf{H}} |u(z)|(1 + |z|^n) < \infty.$$

Then it is shown in [8] that the space $\mathcal{R}$ and D_0 are dense in b^p for all range of $1 < p < \infty$. With these, Ramey and Yi showed in [8] that the dual of b^p can be identified with $b^{p'}$ under the standard integral pairing for $1 < p < \infty$. More precisely, given a bounded linear functional Λ on b^p, there exists a unique function $v \in b^{p'}$ such that

$$\Lambda(u) =< u, v >= \int_{\mathbf{H}} u(w)\overline{v(w)}\, dw.$$

Moreover $\|\Lambda\| \approx \|v\|_{p'}$. Therefore if $u \in b^p$, then $\|u\|_p \approx \sup | < u, v > |$ where supremum is taken over all $v \in D_0$ with $\|v\|_{p'} \leq 1$.

Now we are going to mention some basic properties about pseudohyperbolic balls on $\mathbf{H}$. The pseudohyperbolic distance between two points

$z, w \in \mathbf{H}$ is defined by

$$\rho(z,w) = \frac{|z-w|}{|z-\overline{w}|}.$$

In [4], Choe and Yi showed that ρ is an actual distance which is horizontal translation invariant and dilation invariant. In particular,

$$\rho(z,w) = \rho(\phi_a(z), \phi_a(w))$$

for each $a, z, w \in \mathbf{H}$, where ϕ_a is a map given by

$$\phi_a(z) = \left(\frac{z'-a'}{a_n}, \frac{z_n}{a_n}\right). \tag{2.2}$$

For $z \in \mathbf{H}$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball of radius δ centered at z. Then from the invariant property, we can check easily that $\phi_z(E_\delta(z)) = E_\delta(z_0)$ and we can also check that

$$E_\delta(z) = B\left(\left(z', \frac{1+\delta^2}{1-\delta^2}z_n\right), \frac{2\delta}{1-\delta^2}z_n\right) \tag{2.3}$$

where $B(a,r)$ denotes the euclidean ball of radius r centered at a. ¿From this we have

$$\frac{1-\delta}{1+\delta} < \frac{z_n}{w_n} < \frac{1+\delta}{1-\delta} \tag{2.4}$$

whenever $\rho(z,w) < \delta$.

Now we are going to prove a sequence of lemmas.

Lemma 2.1. *Let $z \in \mathbf{H}$. Then $\|R(z,w)\|_p \approx z_n^{-n}$ for $w \in E_{1/6}(z)$.*

Proof. The inequality $R(z,w) \le C z_n^{-n}$ follows easily from (2.1). Assume $\rho(z,w) = r < 1/6$. Then by (2.3), we have

$$|z - \overline{w}| \le z_n + \frac{1+r}{1-r} z_n < 3z_n.$$

Therefore we have

$$\frac{z_n + w_n}{|z-\overline{w}|} \ge \frac{1-r}{1+r} > 5/7$$

from (2.4) and so

$$
\begin{aligned}
R(z,w) &= \frac{C}{|z-\overline{w}|^n}\left(n\left(\frac{z_n+w_n}{|z-\overline{w}|}\right)^2 - 1\right) \\
&\geq C\ \frac{2(5/7)^2-1}{|z-\overline{w}|^n} \\
&\geq C\ z_n^{-n}.
\end{aligned}
$$

This completes the proof. □

LEMMA 2.2. *If $1<p<\infty$, then $\|R(z,\cdot)\|_p \approx z_n^{-n/p'}$.*

Proof. First note that $R(z,w) = z_n^{-n}(R_{z_0}\circ\phi_z)(w)$ for all $z,w\in\mathbf{H}$, where ϕ_z is a map defined in (2.2). Therefore

$$
\begin{aligned}
\|R(z,\cdot)\|_p^p &= \int_{\mathbf{H}} |R(z,w)|^p\,dw \\
&= z_n^{-np}\int_{\mathbf{H}} |R(z_0,\phi_z(\phi_z^{-1}(w))|^p z_n^n\,dw \\
&= z_n^{-np/p'}\|R(z_0,\cdot)\|_p^p,
\end{aligned}
$$

because the Jacobian of ϕ_z^{-1} is z^n. Hence we get the desired result. □

The following lemma is in [4]

LEMMA 2.3. *Let $0<\delta<1/6$. Then there is a sequence $\{z_m\}$ in $\mathbf{H}$ such that*

$$\cup_{m=1}^{\infty} E_\delta(z_m) = \mathbf{H}$$

and for some $N\in\mathbf{N}$, $E_{2\delta}(z_m)$ intersects at most N of $\{E_{2\delta}(z_k) : k\in\mathbf{N}\}$.

For the next lemma and elsewhere, $|E| = V(E)$ denotes the volume of Borel sets E in $\mathbf{H}$.

LEMMA 2.4. *Let $1<p<\infty$, $0<\delta<1/6$ and let $u\in b^p$. Then there is a positive constant $C = C_{n,p,\delta}$ such that for $a\in\mathbf{H}$*

$$\sup_{z\in E_\delta(a)} |u(z)|^p \leq \frac{C}{|E_\delta(a)|}\int_{E_{2\delta}(a)} |u(w)|^p\,dw.$$

Proof. Let $a\in\mathbf{H}$ and let $z\in E_\delta(a)$. Because $\rho(z,a)<\delta$, we have $z_n\approx a_n$ and

$$B\left(z,\frac{2\delta z_n}{1+\delta}\right)\subset E_\delta(z)\subset E_{2\delta}(a)$$

Therefore from the mean value property and Jenson's inequality, we get

$$
\begin{aligned}
|u(z)|^p &\leq \frac{1}{|B(z, 2\delta z_n/(1+2\delta))|} \int_{B(z,2\delta z_n/(1+2\delta))} |u(w)|^p \, dw. \\
&\leq \frac{C}{z_n^n} \int_{E_{2\delta}(a)} |u(w)|^p \, dw. \\
&\approx \frac{C}{a_n^n} \int_{E_{2\delta}(a)} |u(w)|^p \, dw. \\
&\leq \frac{C}{|E_\delta(a)|} \int_{E_{2\delta}(a)} |u(w)|^p \, dw.
\end{aligned}
$$

The proof is complete. □

3. Main Results

In this section we give characterizations for the Toeplitz operator on b^p for $1 < p < \infty$ with symbol $\mu \in M^+$ to be bounded. In order to do so, we need the notion of Carleson measures. Let $1 < p < \infty$ and let $\mu \in M^+$. Then μ is called a Carleson measure on b^p if the inclusion map $i : b^p \to L^p(\mu)$ is bounded. Namely, μ is a *Carleson measure* on b^p if and only if

$$\int_{\mathbf{H}} |u(w)|^p \, d\mu(w) \leq C_p \int_{\mathbf{H}} |u(w)|^p \, dw$$

for all $u \in b^p$. We refer more on Carleson measures to [1] and [5].

If $\mu \in M^+$ is a Carleson measure on b^p, $u \in \mathcal{R}$ and $v \in D_0$, then it is not hard to check that

$$< T_\mu u, v >= \int_{\mathbf{H}} u(w)\overline{v(w)} \, d\mu(w), \tag{3.1}$$

which enables us to make a connection between Carleson measures and positive Toeplitz operators.

Now we are ready to prove the main result of this paper.

THEOREM 3.1. *Let $1 < p < \infty$, $0 < \delta < 1/6$ and let $\mu \in M^+$. Assume $\{z_m\}$ is a sequence in Lemma 2.3. Then the following conditions are equivalent.*

(1) *μ is a Carleson measure on b^p;*
(2) *$\mu(E_\delta(z))/|E_\delta(z)|$ is bounded for $z \in \mathbf{H}$;*
(3) *$\mu(E_\delta(z_m))/|E_\delta(z_m)|$ is bounded for $m \in \mathbf{N}$;*

(4) T_μ *is bounded on* b^p.

Proof of (1) $\Rightarrow$ (2). Suppose that there is a positive constant $C = C_{n,p}$ such that

$$\int_{\mathbf{H}} |u(w)|^p \, d\mu(w) \le C \int_{\mathbf{H}} |u(w)|^p \, dw. \tag{3.2}$$

Fix $z \in \mathbf{H}$. ¿From (3.2) and Lemma 2.2, we have

$$\int_{\mathbf{H}} |u(w)|^p \, d\mu(w) \le C z_n^{-np/p'}. \tag{3.3}$$

We also have from Lemma 2.1,

$$\int_{\mathbf{H}} |u(w)|^p \, d\mu(w) \ge \int_{E_\delta(z)} |u(w)|^p \, d\mu(w) \ge C \frac{\mu(E_\delta(z))}{z_n^{np}}. \tag{3.4}$$

Therefore from (3.3) and (3.4), we have

$$\frac{\mu(E_\delta(z))}{|E_\delta(z)|} \approx \frac{\mu(E_\delta(z))}{z_n^n} = \frac{\mu(E_\delta(z))}{z_n^{np-np/p'}} \le C,$$

as desired. So the proof is complete. □

Proof of (2) $\Rightarrow$ (3). This implication is clear. □

Proof of (3) $\Rightarrow$ (1). Suppose that there exists a positive constant M such that

$$\sup_{m \in \mathbf{N}} \frac{\mu(E_\delta(z_m))}{|E_\delta(z_m)|} < M.$$

Let $u \in b^p$. Then from Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned}
\int_{\mathbf{H}} |u(z)|^p \, d\mu(z) &= \int_{\cup_{m\in\mathbf{N}} E_\delta(z_m)} |u(z)|^p \, d\mu(z) \\
&\leq \sum_{m=1}^{\infty} \int_{E_\delta(z_m)} |u(z)|^p \, d\mu(z) \\
&\leq \sum_{m=1}^{\infty} \mu(E_\delta(z_m)) \sup_{z\in E_\delta(z_m)} |u(z)|^p \\
&\leq C \sum_{m=1}^{\infty} \frac{\mu(E_\delta(z_m))}{|E_\delta(z_m)|} \int_{E_{2\delta}(z_m)} |u(w)|^p \, dw \\
&\leq CM \sum_{m=1}^{\infty} \int_{E_{2\delta}(z_m)} |u(w)|^p \, dw \\
&\leq CMN \int_{\mathbf{H}} |u(w)|^p \, dw.
\end{aligned}$$

This shows that μ is a Carleson measure on b^p and the proof is complete. $\square$

Remark: So far we proved that (1), (2), (3) are equivalent conditions. Therefore if $\mu \in M^+$ is a Carleson measure on b^{p_0} for some $1 < p_0 < \infty$, then it is a Carleson measure on b^p for every range of $1 < p < \infty$.

Proof of (3) $\Rightarrow$ (4). Suppose (3). Then we know that μ is a Carleson measure on b^p and $b^{p'}$ by the above remark. To complete the proof of this step, we only need to show that T_μ is bounded on $\mathcal{R}$, because $\mathcal{R}$ is a dense subspace of b^p. Let $u \in \mathcal{R}$ and $v \in D_0$. Then from (3.1) and Hölder's inequality, we have

$$\begin{aligned}
| < T_\mu u, v > | &= \left| \int_{\mathbf{H}} u(w)\overline{v(w)} \, d\mu(w) \right| \\
&\leq \left(\int_{\mathbf{H}} |u(w)|^p \, d\mu(w) \right)^{1/p} \left(\int_{\mathbf{H}} |v(w)|^{p'} \, d\mu(w) \right)^{1/p'} \\
&\leq C \left(\int_{\mathbf{H}} |u(w)|^p \, dw \right)^{1/p} \left(\int_{\mathbf{H}} |v(w)|^{p'} \, dw \right)^{1/p'} \\
&= C\|u\|_p \|v\|_{p'}.
\end{aligned}$$

Therefore $T_\mu u \in b^p$ with $\|T_\mu u\|_p \leq C\|u\|_p$ for some $C = C_{n,p}$. So T_μ is bounded on $\mathcal{R}$ as desired. The proof is complete $\square$

Proof of (4) $\Rightarrow$ (2). Assume T_μ is bounded on b^p. Let $z \in \mathbf{H}$. Because $R(z,\cdot) \in \mathcal{R} \cap D_0$, we have

$$\left| \left\langle T_\mu \frac{R(z,\cdot)}{\|R(z,\cdot)\|_p}, \frac{R(z,\cdot)}{\|R(z,\cdot)\|_{p'}} \right\rangle \right| \leq \|T_\mu\| < \infty.$$

On the other hand, from Lemma 2.2 and Lemma 2.1 we get

$$\begin{aligned} \left| \left\langle T_\mu \frac{R(z,\cdot)}{\|R(z,\cdot)\|_p}, \frac{R(z,\cdot)}{\|R(z,\cdot)\|_{p'}} \right\rangle \right| &= \frac{| < T_\mu R(z,\cdot), R(z,\cdot) > |}{\|R(z,\cdot)\|_p \|R(z,\cdot)\|_{p'}} \\ &\geq C\, z_n^n \int_{\mathbf{H}} R(z,w)^2 \, d\mu(w) \\ &\geq C\, z_n^n \int_{E_\delta(z)} R(z,w)^2 \, d\mu(w) \\ &\geq C \int_{E_\delta(z)} \frac{1}{z_n^n} \, d\mu(w) \\ &\approx C \, \frac{\mu(E_\delta(z))}{|E_\delta(z)|}. \end{aligned}$$

Therefore

$$\sup_{z \in \mathbf{H}} \frac{\mu(E_\delta(z))}{|E_\delta(z)|} \leq C \, \|T_\mu\| < \infty,$$

as desired. This completes the proof. $\square$

Remark: From the above theorem we also see that if T_μ is bounded on b^{p_0} for some $1 < p_0 < \infty$, then it is always bounded on b^p for every range of $1 < p < \infty$.

References

[1] K. R. M. Attele, *Analytic multipliers of Bergman spaces*, Michigan Math. J. **30**(1984), 307–319.

[2] S. Axler, *Bergman spaces and their Operators*, Surveys of Some Results in Operator Theory, vol. 1, Pitman Res. Notes Math. Ser. vol. 171, 1988, pp. 1–50.

[3] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer-Verlag, New York, 1992.

[4] B. R. Choe and H. Yi, *Representations and interpolations of harmonic Bergman functions on half spaces*, Nagoya Math. J.**151**(1998), 51–89.

[5] J. Cima and W. Wogen, *A Carleson measure theorem for the Bergman space on the ball*, J. Operator Theory **7**(1982), 157–165.
[6] D. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. **73**(1987), 345–368.
[7] J. Miao, *Toeplitz operators on harmonic Bergman space*, Integr. Equ. Oper. Theory **27**(1997) 426–438.
[8] W. Ramey and H. Yi, *Harmonic Bergman functions on half-spaces*, Trans. Amer. Math. Soc. **348**(1996), 633-660.
[9] K. Zhu, *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains*, J. Operator Theory **20**(1988), 329–357.

Department of Mathematics
Research Institute of Basic Sciences
Kwangwoon University
Seoul 139–701, Korea
E-mail: hsyi@math.kwangwoon.ac.kr