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ON THE NONHOLONOMIC FRAMES IN A GENERALIZED 2-DIMENSIONAL RIEMANNIAN MANIFOLD X_2

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Abstract. We will derive some identities related to p-scalars, p-vectors, and non-holonomic frames in a generalized 2-dimensional Riemannian manifold X_2.

1. Introduction

Let X_2 be a generalized 2-dimensional Riemannian manifold referred to a real coordinate system x^μ_λ and endowed with a real quadratic tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{1.1}$$

In this paper, we assume that

$$\mathfrak{g} = Det((g_{\lambda\mu})) \ , \quad \mathfrak{h} = Det((h_{\lambda\mu})) < 0 \ , \quad \mathfrak{k} = Det((k_{\lambda\mu})) \tag{1.2}$$

According to (1.2), there exists a unique symmetric tensor $h_{\lambda\mu}$ defined by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu \tag{1.3}$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising or lowering indices of tensor quantities in X_2 in the usual manner.

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The purpose of the present paper is, in the first, to study the properties of the p-scalars H and the corresponding eigenvectors P^ν in X_2, defined by

$$(1.4) \qquad (Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})P^\nu = 0, \qquad (H : \text{a scalar})$$

Finally, we introduce a nonholonomic frame to X_2 and find expressions for the tensor $h_{\lambda\mu}$ and ${}^{(p)}k_{\nu\lambda}$ in terms of the p-vectors P^ν.

2. Preliminaries

This section is a brief collection of definitions, notations, and basic results which are needed in our subsequent considerations.

$$(2.1) \qquad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \qquad k = \frac{\mathfrak{k}}{\mathfrak{h}}$$

$$(2.2) \qquad \mathfrak{g} = \mathfrak{h} + \mathfrak{k}$$

$$(2.3) \qquad {}^{(0)}k_\lambda^\nu = \delta_\lambda^\nu, \; {}^{(p)}k_\lambda^\nu = {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu \qquad (p = 1, 2, \cdots)$$

$$(2.4) \qquad \mathfrak{k} = \Omega^2 > 0, \; k = \frac{\Omega^2}{\mathfrak{h}} < 0 \qquad \text{where} \quad \Omega = k_{12}$$

$$(2.5a) \qquad {}^{(p)}k_{\lambda\mu} = (-k)^{\frac{p}{2}} h_{\lambda\mu} \quad \text{if p is even}$$

$$(2.5b) \qquad {}^{(p)}k_{\lambda\mu} = (-k)^{\frac{p-1}{2}} k_{\lambda\mu} \quad \text{if p is odd}$$

Furthermore, we use $\mathcal{E}^{\lambda\mu}(\mathfrak{e}_{\lambda\mu})$ as the contravariant (covariant) indicators of weight $1(-1)$.

The eigenvalue M and the corresponding eigenvector a^ν in X_2 defined by

$$(2.6) \qquad (Mh_{\nu\lambda} - k_{\nu\lambda})a^\nu = 0 \qquad (M : \text{a scalar})$$

are called basic scalars and basic vectors of X_2, respectively.

There are exactly two linearly independent basic vectors a^ν satisfying (2.6), where the corresponding basic scalars M are given by

$$(2.7) \qquad M = M_1 = -M_2 = \sqrt{-k}$$

3. P - Scalars

LEMMA 3.1. *We have*

$$\mathfrak{k} > 0 \quad , \quad k = \frac{\mathfrak{k}}{\mathfrak{h}} < 0 \tag{3.1}$$

Proof. Since $k_{\lambda\mu}$ is skew symmetric, $\mathfrak{k} = (k_{12})^2 = \Omega^2 > 0$ □

LEMMA 3.2. *We have*

$$\begin{aligned} A(H) &\stackrel{def}{=} Det((Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})) \\ &= \begin{cases} \mathfrak{h}(H - (-k)^{\frac{p}{2}})^2, & \textit{if p is even} \\ \mathfrak{h}(H^2 + k^p), & \textit{if p is odd.} \end{cases} \end{aligned} \tag{3.2}$$

Proof. (case 1) p is even:

$$\begin{aligned} 2A(H) &= 2Det((Hh_{\nu\lambda} - {}^{(p)}k_{\mu\beta})) \\ &= \mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}(Hh_{\omega\alpha} - {}^{(p)}k_{\omega\alpha})(Hh_{\mu\beta} - {}^{(p)}k_{\mu\beta}) \\ &= 2H^2\mathfrak{h} - 2H\mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}h_{\omega\alpha}{}^{(p)}k_{\mu\beta} + 2Det(({}^{(p)}k_{\lambda\mu})) \\ &= 2H^2\mathfrak{h} - 2H(2(-k)^{\frac{p}{2}}\mathfrak{h}) + 2k^2\mathfrak{h} \\ &= 2\mathfrak{h}(H - (-k)^{\frac{p}{2}})^2 \end{aligned}$$

(case 2) p is odd:

$$\begin{aligned} 2A(H) &= 2Det((Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})) \\ &= \mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}(Hh_{\omega\alpha} - {}^{(p)}k_{\omega\alpha})(Hh_{\mu\beta} - {}^{(p)}k_{\mu\beta}) \\ &= 2H^2\mathfrak{h} - 2H\mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}h_{\omega\alpha}{}^{(p)}k_{\mu\beta} + 2Det(({}^{(p)}k_{\lambda\mu})) \\ &= 2\mathfrak{h}(H^2 + k^p) \end{aligned}$$

□

THEOREM 3.3. *The eigenvalues H of (2.4) are given by*

$$\begin{cases} H = (-k)^{\frac{p}{2}}, & \text{p is even} \\ H = \pm(-k)^{\frac{p}{2}}, & \textit{p is odd} \end{cases} \tag{3.3}$$

Proof. The existence of the above case is clear from Lemma(3.1). A necessary and sufficient condition for the existence of a nontrivial solution P^ν of (1.4) is $A(H) = 0$. According to Lemma(3.2), the result follows directly. □

Remark. In case that p is even, $H = (-k)^{\frac{P}{2}}$ is a double root of $A(H) = 0$.

Theorem 3.4. *Every basic vector* $\underset{i}{a}^\nu$ *of* X_2 *is also p-vector of* X_2.

Proof. (case 1) p is even:
From (2.5a), we have

$$^{(p)}k_{\nu\lambda}\underset{i}{a}^\nu = (-k)^{\frac{p}{2}} h_{\nu\lambda}\underset{i}{a}^\nu = M^p h_{\nu\lambda}\underset{i}{a}^\nu$$

where $M = \sqrt{-k}$.
(case 2) p is odd:
From (2.5b), we have

$$^{(p)}k_{\nu\lambda}\underset{i}{a}^\nu = (-k)^{\frac{p-1}{2}} k_{\nu\lambda}\underset{i}{a}^\nu = (-k)^{\frac{p-1}{2}} M h_{\nu\lambda}\underset{i}{a}^\nu = M^p h_{\nu\lambda}$$

where $M = M_1 = M_2 = \sqrt{-k}$
In both cases, we have

$$^{(p)}k_{\nu\lambda}\underset{i}{a}^\nu = H h_{\nu\lambda}\underset{i}{a}^\nu$$

where $H = M^p$.
Therefore, $\underset{i}{a}^\nu$ is the p-vector P^ν with p - scalar H. □

Theorem 3.5. *In case that p is even, every vector of* X_2 *is a p-vector of* X_2.

Proof. From (2.5a), the relation (1.4) may be written as

$$(H - (-k)^{\frac{p}{2}}) h_{\nu\lambda} P^\nu = 0 \tag{3.4}$$

Since $\mathfrak{h} \neq 0$, (3.4) holds for every vector P^ν with $H = (-k)^{\frac{p}{2}}$. □

4. P-Vectors

In this section, we shall study and derive several properties of p-vectors given by (1.4).

THEOREM 4.1. *In case that p is odd, the p-vector P^ν is a null vector.*

Proof. Since the tensor ${}^{(p)}k_{\lambda\mu}$ is skew-symmetric and from (1.4), for $i = 1, 2$,

$$H h_{\nu\lambda} \underset{i}{P^\nu} \underset{i}{P^\lambda} = {}^{(p)}k_{\nu\lambda} \underset{i}{P^\nu} \underset{i}{P^\lambda} = -{}^{(p)}k_{\lambda\nu} \underset{i}{P^\lambda} \underset{i}{P^\nu} = -{}^{(p)}k_{\nu\lambda} \underset{i}{P^\nu} \underset{i}{P^\lambda} = 0.$$

In virtue of (3.3), $H \neq 0$, we have

$$h_{\nu\lambda} \underset{i}{P^\nu} \underset{i}{P^\lambda} = 0.$$

□

THEOREM 4.2. *In case that p is odd, there are only two eigenvectors $\underset{i}{P^\nu}$ and $\underset{i}{P^\lambda}$ which have the following properties.*

(a)They are defined up to arbitrary factor of proportionality.
(b)They are null.
(c)They are not perpendicular.
(d)Their directions are real.

Proof. We are only to prove (c).

$$H_2 \underset{2}{P^\nu} \underset{1}{P_\nu} = {}^{(p)}k_{\mu\lambda} \underset{1}{P^\lambda} \underset{2}{P^\mu} = -{}^{(p)}k_{\mu\lambda} \underset{1}{P^\mu} \underset{2}{P^\lambda} = -{}^{(p)}k_\mu^\nu \underset{1}{P^\mu} \underset{2}{P_\mu} = -H_1 \underset{1}{P^\nu} \underset{2}{P_\nu}$$

where $H_1 = (-k)^{\frac{p}{2}}$, $H_2 = -(-k)^{\frac{p}{2}}$. This is equivalent to

$$(H_1 + H_2) h_{\mu\lambda} \underset{1}{P^\mu} \underset{2}{P^\lambda} = 0.$$

Since $H_1 + H_2 = 0$, we have $h_{\mu\lambda} \underset{1}{P^\mu} \underset{2}{P^\lambda}$ is arbitrary, so that $h_{\mu\lambda} \underset{1}{P^\mu} \underset{2}{P^\lambda} \neq 0$. □

THEOREM 4.3. *If p is even, P^ν is not necessarily null.*

Proof.

$$Hh_{\nu\lambda}P^{\nu}P^{\lambda} = {}^{(p)}k_{\nu\lambda}P^{\nu}P^{\lambda} = (-k)^{\frac{p}{2}}h_{\nu\lambda}P^{\nu}P^{\lambda}$$

so that

$$(H - (-k)^{\frac{p}{2}})h_{\nu\lambda}P^{\nu}P^{\lambda} = 0.$$

But

$$(H - (-k)^{\frac{p}{2}}) = 0.$$

Therefore, we have

$$h_{\nu\lambda}P^{\nu}P^{\lambda} \neq 0. \qquad \square$$

5. Nonholonomic Frames in X_2

For our further discussions we make the following agreement.

AGREEMENT. *The factor of proportionality mentioned in Theorem (4.2)a may be chosen in such a way that*

$$h_{\lambda\mu}\underset{1}{P}^{\lambda}\underset{2}{P}^{\mu} = 1. \tag{5.1}$$

Since $\underset{1}{P}^{\lambda}$ and $\underset{2}{P}^{\mu}$ are linearly independent, there exists a unique reciprocal set of two linearly independent covariant vectors $\overset{1}{P}_{\lambda}$ and $\overset{2}{P}_{\lambda}$ such that

$$\underset{i}{P}^{\nu}\overset{i}{P}_{\lambda} = \delta^{\nu}_{\lambda}. \tag{5.2a}$$

These equations are equivalent to

$$\underset{j}{P}^{\lambda}\overset{j}{P}_{\lambda} = \delta^{i}_{j}. \tag{5.2b}$$

DEFINITION 5.1. With the vectors $\underset{i}{P}^\nu$ and $\overset{i}{P}_\lambda$, a nonholonomic frame of X_2 is defined in the following usual way : If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$T_{j\cdots}^{i\cdots} \overset{def}{=} T_{\lambda\cdots}^{\nu\cdots}\overset{j}{P}_\nu \underset{i}{P}^\lambda \cdots \tag{5.3a}$$

An easy inspection of (5.2)a and (5.1) shows

$$T_{\lambda\cdots}^{\nu\cdots} \overset{def}{=} T_{j\cdots}^{i\cdots} P_i^\nu P_\lambda^j \cdots \tag{5.3b}$$

THEOREM 5.1. *Let p be odd. Then the nonholonomic components*

$$h_{ij} = h_{\lambda\mu}\underset{i}{P}^\lambda \underset{j}{P}^\mu \quad ; \quad h^{ij} = h^{\lambda\mu}\overset{i}{P}_\lambda \overset{j}{P}_\mu \tag{5.4}$$

are given by the matrix equation

$$((h_{ij})) = ((h^{ij})) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.5}$$

Proof. From Theorem(4.2b) and (5.1), we have

$$h_{ij}h^{kj} = h_{\mu\lambda}\underset{i}{P}^\mu \underset{j}{P}^\lambda h^{\nu\omega}\overset{k}{P}_\nu \overset{j}{P}_\omega = h_{\mu\lambda}h^{\nu\lambda}\underset{i}{P}^\mu \overset{k}{P}_\nu = \underset{i}{P}^\mu \overset{k}{P}_\nu \delta_\mu^\nu = \delta_i^k . \square \tag{5.6}$$

THEOREM 5.2. *Denote $\mathcal{A}$ and $\mathfrak{a}$ the determinants of $\overset{i}{P}_\lambda$ and $\underset{i}{P}^\nu$, then*

$$\mathcal{A}\mathfrak{a} = 1 \tag{5.7}$$

$$\mathcal{A} = -\gamma\sqrt{\mathfrak{h}} \tag{5.8a}$$

$$\mathfrak{a} = -\gamma/\sqrt{\mathfrak{h}} \qquad \textit{where } \gamma = sgn\mathcal{A} \tag{5.8b}$$

Proof. We have first

$$\mathfrak{a} = \mathfrak{e}_{\omega\mu} \underset{1}{P}^{\omega} \underset{2}{P}^{\mu},$$

$$\mathcal{A} = \mathcal{E}^{\omega\mu} \overset{1}{P}_{\omega} \overset{2}{P}_{\mu}.$$

On the other hand,

$$Det((h_{ij})) = \mathfrak{a}^2 \mathfrak{h} = -1,$$

$$Det((h^{ij})) = \mathcal{A}^2 \mathfrak{h}^{-1} = -1.$$ □

THEOREM 5.3. *The nonholonomic components of indicators are*

(5.9a) $$\mathcal{E}^{ij} = 2!\delta^{ij}_{[12]},$$

(5.9b) $$\mathfrak{e}_{ij} = 2!\delta^{[12]}_{ij}.$$

THEOREM 5.4. *For even p, the nonholonomic components of* ${}^{(p)}k_{\lambda}^{\nu}$ *are*

(5.10) $${}^{(p)}k_x^i = (\underset{x}{M})^p \delta_x^i$$

so that

$${}^{(p)}k_{xi} = (\underset{x}{M})^p h_{xi}.$$

Proof. Our starting equation is equivalent to (1.4), i.e.

$$(\underset{x}{M})^p \underset{x}{P}^{\nu} = {}^{(p)}k_{\mu}^{\nu} \underset{x}{P}^{\mu}.$$

Multiplying this equation by $\overset{i}{P}_{\nu}$ and from (5.2)b,

$$\underset{x}{M}^p \delta_x^i = {}^{(p)}k_{\mu}^{\nu} \underset{x}{P}^{\mu} \overset{i}{P}_{\nu} = {}^{(p)}k_x^i$$ □

THEOREM 5.5. *For even p, the tensors $h_{\lambda\mu}$, ${}^{(p)}k_{\lambda\mu}$, and ${}^{(p)}k^{\lambda\nu}$ may be expressed as follows:*

(5.11a) $$h_{\lambda\mu} = 2\underset{1}{P}_{(\lambda}\underset{2}{P}_{\mu)} \quad , \quad h^{\lambda\nu} = 2\underset{1}{P}^{(\lambda}\underset{2}{P}^{\nu)}$$

(5.11b) $${}^{(p)}k_{\lambda\mu} = 2M^p\overset{1}{P}_{[\lambda}\overset{2}{P}_{\mu]} \ , {}^{(p)}k^{\lambda\nu} = 2M^p\underset{2}{P}^{[\lambda}\underset{1}{P}^{\nu]}$$

Proof.

$$h_{\lambda\mu} = h_{ij}\overset{i}{P}_\lambda\overset{j}{P}_\mu = \overset{1}{P}_\lambda\overset{2}{P}_\mu + \overset{2}{P}_\lambda\overset{1}{P}_\mu = \underset{2}{P}_\lambda\underset{1}{P}_\mu + \underset{1}{P}_\lambda\underset{2}{P}_\mu = 2\underset{2}{P}_{(\lambda}\underset{1}{P}_{\mu)}$$

$${}^{(p)}k_{\lambda\mu} = \underset{x}{M}^p h_{xi} = \underset{x}{M}^p 2\underset{2}{P}_{(\lambda}\underset{1}{P}_{\mu)}$$

in virtue of (5.10) and (5.11)a. □

THEOREM 5.6.

(5.12a) $$\underset{i}{P}^\nu = \overset{j}{P}_\lambda h_{ji}h^{\lambda\nu}$$

(5.12b) $$\overset{j}{P}_\lambda = \underset{i}{P}^\nu h^{ij}h_{\lambda\nu}$$

Proof. Put $X_i^\nu = \overset{j}{P}_\lambda h_{ji}h^{\lambda\nu}$

Multiply this equation by $\overset{i}{P}_\mu$. Then

$$X_i^\nu\overset{i}{P}_\mu = \overset{j}{P}_\lambda\overset{i}{P}_\mu h_{ji}h^{\lambda\nu} = h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu$$

Therefore,

$$X_i^\nu = P_i^\nu$$ □

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