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CONVERGENCE OF EXPONENTIALLY BOUNDED C–SEMIGROUPS

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Abstract. In this paper, we discuss convergence theorem for exponentially bounded C-semigroups. We establish the convergence of the sequence of generators of exponentially bounded C-semigroups in some sense implies the convergence of the sequence of the corresponding exponentially bounded C-semigroups. Under the assumption that $R(C)$ is dense, we show the equivalence between the convergence of generators and exponentially bounded C-semigroups.

1. Introduction

Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ to X. Many problems in applied mathematics occur in the form of a Cauchy problem

$$\text{(CP)} \qquad u'(t) = Au(t),\ t \geq 0, \quad u(0) = x,$$

where $x \in D(A)$. It is well known that for a densely defined linear operator A with nonempty resolvent set, (CP) has a unique solution for all $x \in D(A)$ if and only if A is the infinitesimal generator of a C_0 semigroup $\{T(t) : t \geq 0\}$. And the solution is given by the semigroup, $u(t) = T(t)x$ for every $x \in D(A)$ (see Pazy [6]). For a non-densely defined linear operator A, it is also known that if A is the generator of C-semigroup $\{S(t) : t \geq 0\}$, then (CP) has a unique solution $u(t)$, given by $u(t) = S(t)C^{-1}x$ for all $x \in C(D(A))$ (see deLaubenfels [2] or Tanaka and Miyadera [7]). C-semigroups are a generalization of C_0 semigroups and C-semigroup theory allows us to study many ill-posed Cauchy problems, for example, Schrödinger equation on L^p, $p \neq 2$.

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In the case of C_0 semigroup, the convergence of a sequence of infinitesimal generators in some sense is equivalent to the convergence of the corresponding C_0 semigroups. Roughly speaking, a C_0 semigroup $T(t)$ depends continuously on its infinitesimal generator A and the infinitesimal generator A also depends continuously on the corresponding C-semigroup (see [6]). So the solution of (CP) given by C_0 semigroup depends continuously on A. In this paper, paralleling the convergence of C_0 semigroups we establish the convergence of a sequence of generators of C-semigroups in some sense implies the convergence of a sequence of the corresponding C-semigroups. That is, the solution of (CP) given by C-semigroup also depends on A. Under the assumption that $R(C)$ is dense, we have the equivalence between the convergence of generators and the convergence of corresponding C-semigroups.

2. Convergence of C-semigroups

Throughout this paper, X will be a Banach space. C will be a bounded injective linear operator on X. For an operator A, we will write $D(A)$ for its domain and $R(A)$ for its range.

We start with the definitions and properties of C-semigroup.

Definition 2.1. The family $\{S(t) : t \geq 0\}$ of bounded linear operators from X into itself is called a C-semigroup if it has the following properties;

(1) $S(0) = C$, $S(t + s)C = S(t)S(s)$ for $t,\ s \geq 0$.
(2) For each $x \in X$, $S(t)x$ is continuous in $t \geq 0$.

A C-semigroup $\{S(t) : t \geq 0\}$ is said to be exponentially bounded if there exist $M \geq 0$ and $\omega \in R$ such that $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$.

Definition 2.2. The generator A of a C-semigroup $\{S(t) : t \geq 0\}$ is defined by

$$Ax = C^{-1}\left(\lim_{t\to 0} \frac{1}{t}(S(t)x - Cx)\right)$$

with

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{1}{t}(S(t)x - Cx) \text{ exists and is in } R(C)\}.$$

If $C = I$, the identity operator on X, then a C-semigroup is a C_0 semigroup in the ordinary sense. The infinitesimal generator of a C_0 semigroup is densely defined and has nonempty resolvent set. But the generator of a C-semigroup may not have these properties (see Example 6.2 in [2]). It is not difficult to show that if A is the infinitesimal generator of a C_0 semigroup $\{T(t) : t \geq 0\}$ that commutes with C, then A is the generator of the C-semigroup $\{CT(t) : t \geq 0\}$.

For $r > \omega$, we define a bounded linear operator L_r^A on X by

$$L_r^A x = \int_0^\infty e^{-rt} S(t)x dt \qquad \text{for} \qquad x \in X.$$

Then L_r^A is injective and the linear operator Z defined by

$$Zx = (r - (L_r^A)^{-1} C)x$$

with $D(Z) = \{x \in X : Cx \in R(L_r^A)\}$, is independent of $r > \omega$ (see [1]).

Next, we present some known facts about C-semigroup and its generator, which will be used in the sequel (see [2, 5, 7]).

LEMMA 2.3. *Suppose that A is a generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ satisfying $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Then*

(1) *$A = Z$.*
(2) *A is a closed linear operator with $\overline{D(A)} \supset R(C)$.*
(3) *For all $x \in D(A)$ and $t \geq 0$, $S(t)x \in D(A)$ and $d/dt(S(t)x) = AS(t)x = S(t)Ax$.*
(4) *$r - A$ is injective and $R(C) \subset R(r - A)$ for $r > \omega$.*
(5) *For any $x \in X$, $r > \omega$ and positive integer n, $R(C) \subset D((r - A)^{-n})$ and*

$$(r - A)^{-n} Cx = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-rt} S(t)x dt,$$

which implies $||(r - \omega)^n (r - A)^{-n} C|| \leq M$.

Lemma 2.4. *Suppose that A is the generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ satisfying $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$.*

(1) *For $r > \omega$ and $x \in X$, $L_r^A x \in D(A)$ and $(r - A)L_r^A x = Cx$.*
(2) *For $r > \omega$ and $x \in D(A)$, $L_r^A (r - A)x = Cx$.*

Now we establish the continuous dependence of an exponentially bounded C-semigroup on its generator. Under the assumption that $R(C)$ is dense in X, we can show that the convergence of a sequence of exponentially bounded C-semigroups is equivalent to the convergence of a sequence of their generators. We start with a lemma.

Lemma 2.5. *Let A and B be the generators of the exponentially bounded C-semigroups $\{T(t) : t \geq 0\}$ and $\{S(t) : t \geq 0\}$, respectively, satisfying $||T(t)|| \leq Me^{\omega t}$ and $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Then for every x in X and $r > \omega$*

$$L_r^B (T(t) - S(t))L_r^A x = \int_0^t S(t-s)(L_r^A - L_r^B)T(s)x ds.$$

Proof. Let $x \in X$ and $r > \omega$. Then

$$\begin{aligned}
&\frac{d}{ds}(S(t-s)L_r^B T(s)L_r^A x) \\
&\quad = -S(t-s)BL_r^B T(s)L_r^A x + S(t-s)L_r^B T(s)AL_r^A x \\
&\quad = S(t-s)(C - rL_r^B)T(s)L_r^A x + S(t-s)L_r^B T(s)(rL_r^A - C)x \\
&\quad = CS(t-s)(L_r^A - L_r^B)T(s)x.
\end{aligned}$$

Integrating this equality from 0 to t, we have

$$\begin{aligned}
C\int_0^t S(t-s)(L_r^A - L_r^B)T(s)x ds &= \int_0^t \frac{d}{ds}\left(S(t-s)L_r^B T(s)L_r^A x\right) ds \\
&= S(0)L_r^B T(t)L_r^A x - S(t)L_r^B T(0)L_r^A x \\
&= C(L_r^B T(t)L_r^A x - L_r^B S(t)L_r^A x).
\end{aligned}$$

Since C is injective, the result follows. □

THEOREM 2.6. *Let A and A_n be generators of the exponentially bounded C-semigroups $\{S(t) : t \geq 0\}$ and $\{S_n(t) : t \geq 0\}$, respectively, satisfying $||S(t)|| \leq Me^{\omega t}$ and $||S_n(t)|| \leq Me^{\omega t}$ for $t \geq 0$.*

Suppose that $C(D(A))$ is dense in $R(r - A)$ for $r > \omega$ and

$$\lim_{n\to\infty} (r - A_n)^{-1}y = (r - A)^{-1}y \qquad \text{for } y \in R(C).$$

Then

$$\lim_{n\to\infty} S_n(t)x = S(t)x \qquad \text{for } x \in C(D(A))$$

and the convergence is uniform on bounded t-intervals.

Proof. Suppose that $\lim_{n\to\infty}(r - A_n)^{-1}y = (r - A)^{-1}y$ for all $y \in R(C)$. By Lemma 2.3 (5), we have

$$\lim_{n\to\infty} L_r^{A_n} x = L_r^A x \qquad \text{for } \ x \in X.$$

Let $x \in D(A)$ and let $0 \leq t \leq T$. Then

$$\begin{aligned} S_n(t)Cx - S(t)Cx & \\ &= S_n(t)L_r^A(r - A)x - S(t)L_r^A(r - A)x \\ &= S_n(t)L_r^A(r - A)x - S_n(t)L_r^{A_n}(r - A)x \\ &\quad + S_n(t)L_r^{A_n}(r - A)x - L_r^{A_n}S(t)(r - A)x \\ &\quad + L_r^{A_n}S(t)(r - A)x - S(t)L_r^A(r - A)x \end{aligned}$$

By the similar method in the proof of Theorem 4.2 in [6], the first and third terms in the last equation tend to zero as $n \to \infty$.

By Lemma 2.3 (1), for $z \in D(A)$ there exists $w \in X$ such that $Cz = L_r^A w$. By Lemma 2.5, we have

$$\begin{aligned} L_r^{A_n}S_n(t)Cz - L_r^{A_n}S(t)Cz & \\ &= L_r^{A_n}(S_n(t) - S(t))L_r^A w \\ &= \int_0^t S_n(t - s)(L_r^{A_n} - L_r^A)S(t)wds. \end{aligned}$$

So

$$\begin{aligned}
&||L_r^{A_n} S_n(t)Cz - L_r^{A_n} S(t)Cz|| \\
&\qquad \le \int_0^t ||S_n(t-s)||\ ||(L_r^{A_n} - L_r^{A})S(s)w||ds \\
&\qquad \le \int_0^T ||S_n(t-s)||\ ||(L_r^{A_n} - L_r^{A})S(s)w||ds,
\end{aligned}$$

for $0 \le t \le T$. The integrand is bounded by $2M^3 e^{\omega T}/(r-\omega)||w||$ and it goes to zero as $n \to \infty$. By the Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} (L_r^{A_n} S_n(t)Cz - L_r^{A_n} S(t)Cz) = 0 \quad \text{for } \ z \in D(A),$$

and the convergence is uniform on $0 \le t \le T$.

Since $C(D(A))$ is dense in $R(r-A)$, for given $\varepsilon > 0$ and $x \in D(A)$ there exists Cz in $C(D(A))$ such that $||(r-A)x - Cz|| < \varepsilon$. Thus

$$\begin{aligned}
&||L_r^{A_n} S_n(t)(r-A)x - L_r^{A_n} S(t)(r-A)x|| \\
&\quad \le ||L_r^{A_n} S_n(t)(r-A)x - L_r^{A_n} S_n(t)Cz|| \\
&\quad\quad + ||L_r^{A_n} S_n(t)Cz - L_r^{A_n} S(t)Cz|| \\
&\quad\quad + ||L_r^{A_n} S(t)Cz - L_r^{A_n} S(t)(r-A)x|| \\
&\quad \le \frac{2M^2}{r-\omega} e^{\omega t} ||(r-A)x - Cz|| + ||L_r^{A_n} S_n(t)Cz - L_r^{A_n} S(t)Cz|| \\
&\quad \le \frac{2M^2}{r-\omega} e^{\omega T} \varepsilon + ||L_r^{A_n} S_n(t)Cz - L_r^{A_n} S(t)Cz||
\end{aligned}$$

Therefore $\lim_{n\to\infty} S_n(t)Cx = S(t)Cx$ for all $x \in D(A)$ and the convergence is uniform on bounded t-intervals. □

Under the assumption that $R(C)$ is dense in X, we have the following equivalence between the convergence of generators and the convergence of exponentially bounded C-semigroups.

THEOREM 2.7. *Let A and A_n be generators of the exponentially bounded C-semigroups $\{S(t) : t \geq 0\}$ and $\{S_n(t) : t \geq 0\}$, respectively, satisfying $||S(t)|| \leq Me^{\omega t}$ and $||S_n(t)|| \leq Me^{\omega t}$ for $t \geq 0$.*

Suppose that $R(C)$ is dense in X. Then the following are equivalent:

(1) $\lim_{n\to\infty}(r - A_n)^{-1}y = (r - A)^{-1}y$ *for* $y \in R(C)$.

(2) $\lim_{n\to\infty} S_n(t)x = S(t)x$ *for* $x \in X$ *and the convergence is uniform on bounded t-intervals.*

Proof. Suppose that $\lim_{n\to\infty}(r-A_n)^{-1}y = (r-A)^{-1}y$ for $y \in R(C)$. By Theorem 2.6, we have $\lim_{n\to\infty} S_n(t)Cx = S(t)Cx$ for $x \in D(A)$ and the convergence is uniform on $[0,\ T]$. Since $R(C)$ is dense in X, $D(A)$ is dense and so $C(D(A))$ is also dense in X. The uniform boundedness of $||S_n(t) - S(t)||$ implies

$$\lim_{n\to\infty} S_n(t)x = S(t)x, \quad \text{for } \ x \in X, \quad \text{uniformly on } \ [0,\ T].$$

Assume that (2) holds. For $r > \omega$ and $x \in X$,

$$\begin{aligned} &||(r - A_n)^{-1}Cx - (r - A)^{-1}Cx|| \\ &\qquad = ||\int_0^\infty e^{-rt}S_n(t)xdt - \int_0^\infty e^{-rt}S(t)xdt|| \\ &\qquad \leq \int_0^\infty e^{-rt}||(S_n(t) - S(t))x||dt. \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}(r - A_n)^{-1}Cx = (r - A)^{-1}Cx \qquad \text{for } \ x \in X. \qquad \square$$

If $C = I$, the identity operator on X, then A and A_n are the infinitesimal generators of C_0 semigroups $\{S(t) : t \geq 0\}$ and $\{S_n(t) : t \geq 0\}$, respectively. In this case, $D(A)$ is dense in X and for $r > \omega$, r is in the resolvent set of A and A_n. So the convergence of resolvents on $R(C)$ implies the convergence of resolvents of generators on the whole space X. So our result includes the Trotter-Kato type convergence theorem for C_0 semigroups. And our result also includes Theorem 2.3 in [3]. To see this we only need to show that $\lim_{n\to\infty} A_nx = Ax$ for

$x \in D(A) \subset D(A_n)$ implies that $\lim_{n\to\infty}(r - A_n)^{-1}y = (r - A)^{-1}y$ for $y \in R(C)$. Since $R(C) \subset R(r - A)$, there exists x in $D(A)$ such that $y = (r - A)x$. So we obtain

$$\begin{aligned} &||(r - A_n)^{-1}y - (r - A)^{-1}y|| \\ &\qquad = ||(r - A_n)^{-1}(r - A)x - (r - A)^{-1}(r - A)x|| \\ &\qquad = ||(r - A_n)^{-1}(r - A_n + A_n - A)x - x|| \\ &\qquad = ||A_n x - Ax||. \end{aligned}$$

References

[1] E.B. Davies and M.M.H. Pang, *The Cauchy problem and a generalization of the Hille-Yosida theorem*, Proc. London Math. Soc. **55** (1987), 181 - 248.

[2] R. deLaubenfels, *C-semigroups and the Cauchy problem*, J. Funct. Anal. **111** (1993), 44 - 61.

[3] Young S. Lee, *Convergence of C-semigroups*, Kangweon-Kyungki Math. Jour. **6** (1998), 9 - 15.

[4] Y. Lei and Q. Zheng, *The application of C-semigroups to differential operators in* $L^p(R^n)$, J. Math. Anal. Appl. **188** (1994), 809 - 818.

[5] M. Li and F. Huang, *Characterization of contraction C-semigroups*, Proc. Amer. Math. Soc. **126** (1998), 1063 - 1069.

[6] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

[7] N. Tanaka and I. Miyadera, *Exponentially bounded C-semigroups and integrated semigroups*, Tokyo J. Math. **12** (1989), 99 - 115.

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