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A NOTE ON SIMPLE SINGULAR GP-INJECTIVE MODULES

Sang Bok Nam

Abstract. We investigate characterizations of rings whose simple singular right R-modules are GP-injective. It is proved that if R is a semiprime ring whose simple singular right R-modules are GP-injective, then the center $Z(R)$ of R is a von Neumann regular ring. We consider the condition (*): R satisfies $l(a) \subseteq r(a)$ for any $a \in R$. Also it is shown that if R satisfies (*) and every simple singular right R-module is GP-injective, then R is a reduced weakly regular ring.

1. Introduction

Throughout this paper R denotes an associative ring with identity, and all modules are unitary right R-modules. For any nonempty subset X of a ring R, $r(X)$ and $l(X)$ denote the right annihilator of X and the left annihilator of X, respectively. A right R-module M is called generalized right principally injective (briefly right GP-injective) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M. This concept was introduced by Yuechiming [11]. Von Neumann regularity of rings whose simple right R-modules are injective was studied by many authors in [4], [6], [9] and [10]. Recently, we proved that a ring R is strongly regular iff R is a right quasi-duo ring whose simple right R-modules are GP-injective [6]. Rings whose simple singular right R-modules are injective were studied by many authors in [1], [2] and [8]. Yuechiming proved that a ring R is strongly regular iff R is a semiprime right quasi-duo ring whose simple singular right R-modules are p-injective [10]. Recently, Ding and Chen proved that a ring R is strongly regular iff R is a right duo ring whose simple singular right R-modules are GP-injective [3]. We also proved that a ring R is

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strongly regular iff R is an abelian right quasi-duo ring whose simple singular right R-modules are GP-injective [5].

In this paper we consider rings whose simple singular right R-modules are GP-injective. We investigate characterizations of rings whose every simple singular right R-module is GP-injective. Actually we prove the following facts: **1.** If R is a semiprime ring whose simple singular right R-modules are GP-injective, then the center $Z(R)$ of R is a von Neumann regular ring. **2.** Assume that every simple singular right R-module is GP-injective. If R satisfies (*), then R is a reduced weakly regular ring.

2. Rings whose simple singular modules are GP-injective

We begin with the following lemmas.

Lemma 1.. *If R is a semiprime ring, then $r(a^n) = r(a)$ for any $a \in Z(R)$ and $n \in \mathbb{Z}^+$, where $Z(R)$ denotes the center of R.*

Proof. It can be easily verified by using induction. □

The following lemma is well-known, so we omit its proof.

Lemma 2.. *For any $a \in Z(R)$, if $a = ara$ for some $r \in R$, then there exists $b \in Z(R)$ such that $a = aba$.*

Proposition 3.. *If R is a semiprime ring whose simple singular right R-modules are GP-injective, then the center $Z(R)$ of R is a von Neumann regular ring.*

Proof. First we will show that $aR + r(a) = R$ for any $a \in Z(R)$. If not, there exists a maximal right ideal M of R such that $aR+r(a) \subseteq M$. Since $a \in Z(R)$, $aR+r(a)$ is an essential right ideal and so M must be an essential right ideal of R. Therefore R/M is GP-injective. So there exists a positive integer n such that any R-homomorphism of a^nR into R/M extends to one of R into R/M. Let $f : a^nR \to R/M$ be defined by $f(a^nr) = r + M$. Since R is semiprime, by Lemma 1 f is a well-defined R-homomorphism. Now R/M is GP-injective, so there exists $c \in R$ such that $1 + M = f(a^n) = ca^n + M$. Hence $1 - ca^n \in M$ and so $1 \in M$, which is a contradiction. Therefore $aR + r(a) = R$ for any

$a \in Z(R)$ and so we have $a = ara$ for some $r \in R$. Applying Lemma 2, $Z(R)$ is a von Neumann regular ring. □

Recall that a ring R is right (left) weakly regular if $I^2 = I$ for each right (left) ideal I of R; equivalently, $a \in aRaR\,(a \in RaRa)$ for every $a \in R$. R is weakly regular if it is both right and left weakly regular [7]. Rings whose simple right R-modules are GP-injective are always semiprime [6]. But in general rings whose simple singular right R-modules are injective (hence also GP-injective) need not be semiprime [8].

We consider the condition (*): R satifies $l(a) \subseteq r(a)$ for any $a \in R$.

Lemma 4.. *If R satisfies (*), then $RaR + r(a)$ is an essential right ideal of R.*

Proof. Given $a \in R$, assume that $[RaR + r(a)] \bigcap I = 0$ where I is a right ideal of R. Then $Ia \subseteq I \bigcap RaR = 0$ and so $I \subseteq l(a) \subseteq r(a)$. Hence $I = 0$; whence $RaR + r(a)$ is an essential right ideal of R. □

Lemma 5.. *If R satisfies (*) and every simple singular right R-module is GP-injective, then R is a reduced.*

Proof. Let $a^2 = 0$. Suppose that $a \neq 0$. By Lemma 4, $r(a)$ is an essential right ideal of R. Since $a \neq 0$, $r(a) \neq R$. Thus there exists a maximal essential right ideal M of R containing $r(a)$. Therefore R/M is GP-injective. So any R-homomorphism of aR into R/M extends to one of R into R/M. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. Clearly f is a well-defined R-homomorphism. Thus $1 + M = f(a) = ca + M$. Hence $1 - ca \in M$ and so $1 \in M$, which is a contradiction. Hence $a = 0$, and so R is reduced. □

Theorem 6.. *If R satisfies (*) and every simple singular right R-module is GP-injective, then R is a reduced weakly regular ring.*

Proof. By Lemma 5, R is a reduced ring. We will show that $RaR + r(a) = R$ for any $a \in R$. Suppose that there exists $b \in R$ such that $RbR + r(b) \neq R$. Then there exists a maximal right ideal M of R containing $RaR + r(b)$. By Lemma 4, M must be essential in R. Therefore R/M is GP-injective. So there exists a positive integer n such that any R-homomorphism of $b^n R$ into R/M extends to one of R into R/M. Let

$f : b^n R \to R/M$ be defined by $f(b^n r) = r + M$. Since R is a reduced ring, f is a well-defined R-homomorphism. Now R/M is GP-injective, so there exists $c \in R$ such that $1 + M = f(b^n) = cb^n + M$. Hence $1 - cb^n \in M$ and so $1 \in M$, which is a contradiction. Therefore $RaR + r(a) = R$ for any $a \in R$. Hence R is a right weakly regular ring. Since R is reduced, it is also can be easily verified that R is a weakly regular ring. $\square$

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Department of Mathematics
Kyungdong Univerity
Kosung 219-830, Korea