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A LOWER ESTIMATE OF THE BANACH-MAZUR DISTANCES ON THE QUASI-NORMED SPACES

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Abstract. In this paper we estimate a lower bound of the Banach-Mazur distance between a finite dimensional nonlocally convex space and its Banach envelope space by investigating the properties of the nonlocally convex space and the projection constant which are obtained by factoring the identity operator through l_∞^k on the quasi-normed spaces.

1. Introduction

In this article, we give some lower estimates for the Banach-Mazur distances between finite-dimensional quasi-Banach spaces and its Banach envelope spaces, analogous to known results for Banach spaces.

In 1948, F. John proved that if E is a n-dimensional Banach space then $d(E, l_1^n) \leq \sqrt{n}$ where $dimE = n$. For the quasi-Banach space E, S. J. Dilworth proved that $d(E, l_2^n) \leq n^{1/p-1/2}$ in 1985. (see [1, theorem 2])

Also, if E is a n-dimensional quasi-normed space and E^b its Banach envelope space with norm given by the

$$|\|x\|| = \inf\{\lambda \in \mathbf{R} |\ x \in \lambda co(B_E)\}$$

where $co(B_E) = co\{x \in E |\ \|x\| \leq 1\}$, then

$$d(E, E^b) \leq n^{1/p-1} \tag{1.1}$$

which is due to S. J. Dilworth: see ([1]). In this parallel of question, we can see that if $(E, \|\cdot\|)$ is a $n-$dimensional $p-$convex quasi-normed

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space, then $d(E, l_p^n) \leq n^{1/p}$: see ([7]). Moreover if we restrict conditions on E, we can see a better upper bound of $d(E, l_p^n)$ as following. For $2/3 < p < 1$ there is a constant C_p such that if E is any $n-$dimensional symmetric $p-$convex space, then

$$d(E, l_p^n) \leq C_p n^{2/p-3/2}, \tag{1.2}$$

which is due to Peck : see more details in ([7]).

From now we are concerned with the lower estimates of the Banach-Mazur distance between the quasi-normed space E and its envelope space E^b. For the study of these questions, we need a definition which was given by Peck in ([6]).

Definition 1.1. Let E be a quasi-normed space and let $T : E \longrightarrow l_\infty^k$ be an embedding map and $P : l_\infty^k \longrightarrow E$ be a projection map. Consider the following diagram.

$$\begin{array}{ccc} E & \xrightarrow{Id_E} & E \\ T \downarrow & \nearrow P & \\ l_\infty^k & & \end{array} \tag{1.3}$$

Define

$$\underline{\lambda}(E) = \inf\{\|P\|\|T\| \; | Id_E = PT\},$$

where the infimum is taken over all factorizations $Id_E = PT$ as given above diagram.

When E is $l_p^n, 1 \leq p$, the constant is known: see the references given for that theorem ([8, theorem 32.9 p254]). For $p < 1$, Peck proved that the lower estimate $\underline{\lambda}(l_p^n)$ is $\geq Cn^{1/p-1/2}(\log n)^{-1/2}$ in ([6]).

2. A lower estimates of $d(E, E^b)$.

From these ideas, we will give a lower estimate of $d(E, E^b)$ where E is a quasi-normed space and E^b its envelope space.

Lemma 2.1. *Let E be a $n-$dimensional quasi-Banach space and E^b its Banach envelope space. Then $d(E, E^b) \geq \underline{\lambda}(E)/\underline{\lambda}(E^b)$.*

Proof. Let E be a $n-$dimensional quasi-Banach space and E^b its Banach envelope space and let $T : E \longrightarrow E^b$ be an isomorphism. Consider the following commutative diagrams,

$$\begin{array}{ccc} E^b & \xrightarrow{Id_{E^b}} & E^b \\ T \uparrow & & \downarrow T^{-1} \\ E & \xrightarrow{Id_E} & E \end{array} \tag{2.1}$$

and

$$\begin{array}{ccc} E^b & \xrightarrow{Id_{E^b}} & E^b \\ u_1 \downarrow & \nearrow u_2 & \\ l_\infty^k & & \end{array} \tag{2.2}$$

Then by combining two diagrams(2.1) and (2.2) and by the definition of projection constant $\underline{\lambda}(E)$, we can have

$$\begin{aligned} \underline{\lambda}(E) &= \inf\{\|u_1 T\|\|T^{-1}u_2\| : u_1 : E^b \to l_\infty^k, \quad u_2 : l_\infty^k \to E^b\} \\ &\le \|u_1 T\|\|T^{-1}u_2\| \\ &\le \|u_1\|\|u_2\|\|T\|\|T^{-1}\|. \end{aligned}$$

Next, taking the infimum over all isomorphisms $T : E \longrightarrow E^b$, we have

$$\underline{\lambda}(E) \le \|u_1\|\|u_2\| d(E, E^b).$$

Finally, taking the infimum over all imbeddings $u_1 : E^b \longrightarrow l_\infty^k$ and all projections $u_2 : l_\infty^k \longrightarrow E^b$, we can have

$$\underline{\lambda}(E) \le \underline{\lambda}(E^b) d(E, E^b). \tag{2.3}$$

Therefore we have

$$d(E, E^b) \ge \underline{\lambda}(E)/\underline{\lambda}(E^b).$$

This proves the lemma. □

From this lemma, we will investigate a lower estimate of $d(l_p^n, (l_p^n)^b)$ where $0 < p < 1$. As we knew that the Banach envelope of l_p^n is l_1^n, we see

$$d(l_p^n, (l_p^n)^b) = d(l_p^n, l_1^n). \tag{2.4}$$

For these spaces, we can find a positive function f on $[1, \infty) \times (0, 1)$ such that $d(l_p^n, (l_p^n)^b) \ge cf(n, p)$ where c is a universal constant.

THEOREM 2.2. *Let $0 < p < 1$ and $E = l_p^n$. Then there exists a positive function f on $[1, \infty) \times (0, 1)$ such that $d(l_p^n, (l_p^n)^b) \geq cf(n, p)$ where c is a universal constant.*

Proof. For $0 < p < 1$, as we saw the Banach envelope of l_p^n is l_1^n, and so by (2.4) we have $d(l_p^n, (l_p^n)^b) = d(l_p^n, l_1^n)$.

Now by ([6, corollary 2]) we can see that

$$\underline{\lambda}(E) \geq Cn^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}. \tag{2.5}$$

Also, by the ([3] and [8, theorem 32.8]) we have

$$\underline{\lambda}(l_1^n) = \begin{cases} \leq \sqrt{n} & \text{for complex case,} \\ 2^{-n}\sum_{k=1}^{n}\binom{n}{k}|n-2k| & \text{for the real case.} \end{cases}$$

Hence applying the lemma 2.1, we can have

i) For the complex case

$$\begin{aligned} d(l_p^n, l_1^n) &\geq C\frac{n^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}}{n^{1/2}} \\ &= Cn^{\frac{1}{p}-1}(\log n)^{-\frac{1}{2}}, \end{aligned} \tag{2.6}$$

ii) For the real case,

$$\begin{aligned} d(l_p^n, l_1^n) &\geq C\frac{n^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}}{2^{-n}\sum_{n=1}^{n}\binom{n}{k}|n-2k|} \\ &= Cf(n,p), \end{aligned} \tag{2.7}$$

where $f(n,p) = \frac{n^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}2^n}{2^{-n}\sum_{n=1}^{n}\binom{n}{k}|n-2k|}$. This proves the theorem. □

We don't know yet, whether this lower estimate of the $d(l_p^n, (l_p^n)^b)$ is sharp or not. Nevertheless, we found a function f which depends only on dimension of the given quasi-normed space. We studied typical quasi-normed space l_p^n and its Banach envelope space $(l_p^n)^b$ in the theorem 2.2. Moreover to get a lower estimate of Banach Mazur distance, we used mainly known facts which were the ([6, corollary 2]) and the ([8, theorem 32.8]).

In this parallel of the question we can ask that if E is just a finite dimensional quasi-normed space, does there exist a positive function f on $[1, \infty)$ such that $d(E, E^b) \geq Cf(n)$, where C is a universal constant independent of n ? For this question we can give the partial answer as following.

THEOREM 2.3. *Let E be a n-dimensional quasi-normed space. Then there exists a positive function f on $[1,\infty)$ and also depending on space E such that $d(E,E^b) \geq Cf(n)$ where C is a universal constant independent of n.*

Proof. By the lemma 2.1, we know that $d(E,E^b) \geq \underline{\lambda}(E)/\underline{\lambda}(E^b)$.

Since $(E^b, |\|\cdot\||)$ is a Banach space with dimension n, we knew that the upper bound of $\underline{\lambda}(E^b)$ is

$$\underline{\lambda}(E^b) \leq \sqrt{n} \tag{2.8}$$

in the ([3] and [8, theorem 32.8]). Moreover by the ([6, corollary 1]), we have

$$\underline{\lambda}(E) \geq s_n (n \log n)^{\frac{1}{2}}, \tag{2.9}$$

where $s_n = \sup_{\substack{\|Te_i\| \leq 1 \\ 1 \leq i \leq n}} \inf_{\pm 1} \| \sum_{i=1}^{n} \pm e_i \|$.

Then applying the lemma 2.1, (2.8) and (2.9) we have

$$\begin{aligned} d(E,E^b) &\geq \frac{Cs_n (n \log n)^{\frac{1}{2}}}{\sqrt{n}} \\ &= Cs_n (\log n)^{\frac{1}{2}} \\ &= Cf(n). \end{aligned} \tag{2.10}$$

where $f(n) = s_n (\log n)^{\frac{1}{2}}$. This proves the theorem. □

In the above theorem 2.3, we only have a function f on $[1,\infty)$ which is also depending on the given quasi-normed space E . So the lower estimate of $d(E,E^b)$ is variable depending on the given quasi-Banach space E. To get the better lower estimate, we must overcome the nonlocally convexity of the quasi-normed space E. But this question is still open. For this parallel of question, if we restrict the condition on the quasi-normed space E, we can get a lower estimate of $d(E,E^b)$ which is independent on the given space E. To study this we need a technical lemma.

LEMMA 2.4. *Let $0 < p < 1$ and E be a $p-$convex quasi-normed space. Then the Banach-Mazur distance $d(E, l_p^n) \leq n^{2/p-1}$ and $d(E^b, l_p^n) \leq n^{1/p}$.*

Proof. From the submultiplicative property of the Banach-Mazur distance coefficient, we can have

$$\begin{aligned} d(E, l_p^n) &\leq d(l_p^n, l_2^n) d(l_2^n, E) \\ &\leq n^{1/p-1/2} n^{1/p-1/2} \\ &= n^{2/p-1}. \end{aligned} \tag{2.11}$$

By the same argument as above and using the F. John's theorem [2], we can see

$$d(E^b, l_p^n) \leq n^{1/p}. \tag{2.12}$$

This proves the lemma. □

Now let E be a $p-$convex quasi-normed space. Then there exists a positive function f on $[1, \infty)$ such that $d(E, E^b) \geq f(n)$ where f is depending only on n. Since

$$d(l_p^n, l_p^n) \leq d(l_p^n, E) d(E, E^b) d(E^b, l_p^n).$$

we have

$$\begin{aligned} d(E, E^b) &\geq \frac{1}{d(l_p^n, E) \cdot d(E^b, l_p^n)} \\ &\geq \frac{1}{n^{1/p} \cdot n^{2/p-1}} \\ &= n^{1-3/p}. \end{aligned}$$

Combining the above and the ([1, theorem 2]) , for the $p-$convex quasi-normed space we can have $n^{1-3/p} \leq d(E, E^b) \leq n^{1/p-1/2}$. Though this lower estimate of $d(E, E^b)$ may not be sharp, we can overcome the nonconvexity of the quasi-normed space E.

In this parallel of the question, if we restrict $p = \log 2 / \log 2C$ where C is the quasi-norm constant, we can have a lower estimate of the Banach-Mazur distance $d(E, l_2^n)$ as the following.

THEOREM 2.5. *Let E be a $n-$dimensional quasi-Banach space with quasi-norm constant C and let $p = (\log 2/\log 2C)$. Then we have*

$$d(E, l_2^n) \geq \underline{\lambda}(E)/c_1^n$$

where $c_1^n = \int_{S^{n-1}} |x_1| d\mu(x)$ in the real case and $c_1^n = \int_{\tilde{S}^{n-1}} |z_1| d\tilde{\mu}(z)$ in the complex and $a_1 \leq c_1^n \leq 1$ (a_1 is the first absolute Gaussian Moment).

Proof. By the lemma 2.1 we can have $d(E, l_2^n) \geq \underline{\lambda}(E)/\underline{\lambda}(l_2^n)$. And by the ([8, theorem 32.8]) we see $\underline{\lambda}(l_2^n) = C_1^n\sqrt{n} \sim \sqrt{n}$. Therefore we have

$$\begin{aligned} d(E, l_2^n) &\geq \underline{\lambda}(E)/\underline{\lambda}(l_2^n) \\ &\geq \underline{\lambda}(E)/c_1^n\sqrt{n} \end{aligned}$$

where $a_1 \leq c_1^n \leq 1$ and a_1 is the first absolute Gaussian Moment. This proves the theorem. □

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