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CONFORMAL CHANGE OF THE TENSOR $U^{\nu}{}_{\lambda\mu}$ IN 5-DIMENSIONAL g-UFT

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Abstract. We investigate change of the tensor $U^{\nu}{}_{\lambda\mu}$ induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8],1957). CHUNG([6],1968) also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $^*g$-UFT, and for the second class in 6-dimensional g-UFT were investigated by CHO ([1],1992, [2],1994, [3],1995, [4],1998).

In the present paper, we investigate change of the tensor $U^{\nu}{}_{\lambda\mu}$ induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG([5],1988; [3],1988), CHO([1],1992; [2],1994; [3],1995).

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2.1. n-dimensional g-unified field theory

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([8],1957) and systematically introduced by CHUNG([7],1963).

Let ${X_n}^1$ be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$\text{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor g_{λ_μ} which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part ${k_{\lambda\mu}}^2$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\text{Det}((g_{\lambda\mu})) \neq 0 \qquad \text{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma_{\omega\mu}^\nu$ with the following transformation rule :

$$\Gamma_{\omega'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma_{\beta\gamma}^\alpha + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2 {S_{\omega\mu}}^\alpha g_{\lambda\alpha} \tag{2.6}$$

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[1] Throughout the present paper, we assumed that $n \geq 2$.

[2] Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convention.

where D_ω denotes the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$. The connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

$$\begin{aligned} &\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0, \\ &\mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})), \end{aligned} \tag{2.8a}$$

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \qquad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{2.8b}$$

$$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots) \tag{2.8c}$$

$$^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \qquad ^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \qquad ^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu, \tag{2.8d}$$

$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu}, \tag{2.8e}$$

$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \tag{2.8f}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$K_0 = 1; K_n = k \text{ if } n \text{ is even}; \qquad K_p = 0 \text{ if } p \text{ is odd}, \tag{2.9a}$$

$$g = 1 + K_2 + \cdots + K_{n-\sigma}, \tag{2.9b}$$

(2.9)c $$^{(p)}k_{\lambda\mu} = (-1)^{p(p)} k_{\mu\lambda}, \qquad {}^{(p)}k^{\lambda\nu} = (-1)^{p(p)} k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T:

(2.10)a $$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma} {}^{(p)}k_{\omega}{}^{\alpha} {}^{(q)}k_{\mu}{}^{\beta} {}^{(r)}k_{\nu}{}^{\gamma},$$

(2.10)b $$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)c $$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10)d $$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11) $$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^{\nu}_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12) $$\Gamma^{\nu}_{\omega\mu} = \left\{ {}^{\ \nu}_{\omega\mu} \right\} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu}$$

where

(2.13) $$U_{\nu\omega\mu} = 2\overset{001}{S}{}_{\nu(\omega\mu)}.$$

The above two relations show that *our problem of determining* $\Gamma^{\nu}_{\omega\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

(2.14) $$S = B - 3\overset{(110)}{S}$$

where

(2.15) $$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^{\alpha}k_{\nu}{}^{\beta}.$$

2.2. Some results for the second class in 5-g-UFT

In this section, we introduce some results of 5-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHO([1],1992).

Definition 2.1. In 5-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class , if $K_2 \neq 0$, $K_4 = 0$.

Theorem 2.2. *(Main recurrence relations) For the second class in 5-UFT, the following recurrence relation hold*

$$ {}^{(p+3)}k_{\lambda}{}^{\nu} = -K_2 {}^{(p+1)}k_{\lambda}{}^{\nu}, \qquad (p = 0, 1, 2, \cdots). \tag{2.16}$$

Theorem 2.3. *(For the second class in 5-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$1 - (K_2)^2 \neq 0. \tag{2.17}$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(1 - K_2{}^2)(S - B) = -2 \overset{(10)1}{B} + (K_2 - 1) \overset{110}{B} + 2 \overset{(20)2}{B} + 2 \overset{112}{B}. \tag{2.18}$$

3. Conformal change of the 5-dimensional tensor $U^{\nu}{}_{\lambda\mu}$ for the second class.

In this final chapter we investigate the change $U^{\nu}{}_{\lambda\mu} \rightarrow \overline{U}^{\nu}{}_{\lambda\mu}$ of the tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^{\nu}{}_{\lambda\mu}$. An explicit representation of the change of 5-dimensional tensor $U^{\nu}{}_{\lambda\mu}$ for the second class will be exhibited in this chapter.

Agreement (3.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T ($\overline{T}$) will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\overline{h}_{\lambda\mu})$.

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1998).

Theorem 3.2. *In n-g-UFT, the conformal change (3.1) induces the following changes :*

$$\begin{aligned} &{}^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)} k_{\lambda\mu}, \qquad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \\ &{}^{(p)}\overline{k}^{\lambda\nu} = e^{-\Omega(p)} k^{\lambda\nu} \end{aligned} \tag{3.2a}$$

$$\overline{g} = g, \qquad \overline{K_p} = K_p, \qquad (p = 1, 2, \cdots). \tag{3.2b}$$

Now, we are ready to derive representations of the changes $U^{\nu}{}_{\lambda\mu} \to \overline{U}^{\nu}{}_{\lambda\mu}$ in 5-g-UFT for the second class induced by the conformal change (3.1).

Theorem 3.3. *The change $S_{\lambda\mu}{}^{\nu} \to \overline{S}_{\lambda\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned} \overline{S}_{\lambda\mu}{}^{\nu} = S_{\lambda\mu}{}^{\nu} + \frac{1}{C}[&(3 - K_2 + K_2{}^2){}^{(2)}k^{\nu}{}_{[\lambda}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+ 2K_2{}^2 k^{\nu}{}_{[\lambda}k_{\mu]}{}^{\delta}\Omega_{\delta} + (4K_2 + 2K_2{}^2)k^{\nu}{}_{[\lambda}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+ (1 - K_2 + 4K_2{}^2)k_{\lambda\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta} + (-1 - K_2)k_{\lambda\mu}\Omega^{\nu} \\ &+ (1 + K_2)k^{\nu}{}_{[\lambda}\Omega_{\mu]} + (-1 - K_2{}^2)h^{\nu}{}_{[\lambda}k_{\mu]}{}^{\delta}\Omega_{\delta}] \end{aligned} \tag{3.3}$$

where $C = K_2{}^2 - 1$.

Theorem 3.4. *The change $U^{\nu}{}_{\lambda\mu} \to \overline{U}^{\nu}{}_{\lambda\mu}$ induced by the conformal change (3.1) may be represented by*

$$\begin{aligned} \overline{U}^{\nu}{}_{\lambda\mu} = &U^{\nu}{}_{\lambda\mu} + \frac{1}{C}\{C_1 k^{\nu}{}_{(\lambda}k_{\mu)}{}^{\delta}\Omega_{\delta} \\ &+ C_2[{}^{(2)}k^{\nu}{}_{(\lambda}{}^{(2)}k_{\mu)}{}^{\delta}\Omega_{\delta} + {}^{(2)}k_{\lambda\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta}] \\ &+ C_3[{}^{(2)}k^{\nu}{}_{(\lambda}\Omega_{\mu)} - {}^{(2)}k_{\lambda\mu}\Omega^{\nu}]\} \end{aligned} \tag{3.4}$$

where $C_1 = -6K_2{}^3 + 2K_2{}^2 - K_2 - 1$, $C_2 = 2K_2(K_2 + 2)$, $C_3 = 1 + K_2$.

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$\overline{U}_{\nu\lambda\mu} = \overline{\overset{100}{S}}_{(\lambda\mu)\nu}. \tag{3.5}$$

The relation (3.4) follows by substituting (3.3), (2.10), Definition (2.1), (2.16), (3.2) into (3.5). □

References

[1] C.H. Cho, *Conformal change of the connection in 3- and 5-dimensional $^*g^{\lambda\nu}$-unified Field Theory*, BIBS, Inha Univ. **13** (1992).

[2] C.H. Cho, *Conformal change of the connection for the first class in 5-dimensional $^*g^{\lambda\nu}$-unified Field Theory*, Kangweon-Kyungki Math. Jour. **2** (1994), 19–33.

[3] C.H. Cho, *Conformal change of the torsion tensor in 6-dimensional g-unified field theory*, Kangweon-Kyungki Math. Jour. **3, No. 1** (1995), 71–79.

[4] C.H. Cho, *Conformal change of the tensor $S_{\lambda\mu}{}^{\nu}$ in 5-dimensional g-UFT*, Kangweon-Kyungki Math. Jour. **6** (1998), 213–220.

[5] K.T. Chung, *Three- and Five-dimensional considerations of the geometry of Einsteins's $^*g$-unified field theory*, International Journal of Theoretical Physics **27(9)** (1988).

[6] K.T. Chung, *Conformal change in Einstein's $^*g^{\lambda\nu}$-unified field theory*, Nuove Cimento **(X) 58B** (1968).

[7] K.T. Chung, *Einstein's connection in terms of $^*g^{\lambda\nu}$*, Nuove Cimento **(X) 27** (1963).

[8] V. HLAVATÝ, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd (1957).

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