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t-LINKED OVERRINGS OF A NOETHERIAN DOMAIN

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Abstract. Let R be a Noetherian domain. It is proved that t-dim$R = 1$ if and only if each (proper if R is not a valuation domain) t-linked overring D of R is of t-dim$D = 1$ if and only if each integrally closed t-linked overring of R is a Krull domain.

Throughout this paper, R will be an integral domain with identity having the quotient field K and R' the integral closure of R in K. $X^1(R)$ denotes the set of height one prime ideals of R. An overring of R is a ring between R and K. An overring D of R is t-linked over R if, for each finitely generated fractional ideal A of R such that $A^{-1} = R$, $(AD)^{-1} = D$. If X is a set of prime t-ideals of R, then $\cap_{P \in X} R_P$ is called a subsection of R.

In [[4], Theorem 3.8], B. G. Kang showed that if D is an overring of a Prufer v-multiplication domain (PVMD) R, then D is t-linked over R if and only if D is a subintersection of R. Since a Krull domain is a PVMD, each t-linked overring of a Krull domain D is a subintersection of D and hence a Krull domain. Mori-Nagata Theorem [[6], Theorem 33.10] states that the integral closure of a Noetherian domain is a Krull domain. [[1], Corollary 2.3] says that the integral closure of a Noetherian domain R is t-linked over R. Thus it is natural to ask whether an integrally closed overring of a Noetherian domain is a Krull domain. Unfortunately, this is not true. For example, let R be a local Noetherian domain with maximal P such that ht$P \geq$ 2and P is a t-ideal of R. Let V be a valuation domain with maximal M such that $P = M \cap R$ and ht$P$=ht$M$ (cf. [[2], Corollary 19.7]), then V is t-linked over R but V is not a Krull domain (note that a valuation domain is a Krull domain if and only if it is a rank one DVR).

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In this paper, we give necessary and sufficient conditions for each integrally closed t-linked overring of a Noetherian domain to be a Krull domain. As for background we assume familiarity with t-theoretic notations (cf. [[2], Chap. 32 and 34]). For undefined definitions and notations, the reader can be referred to [[2], [5]].

Now we give the main result of this paper.

THEOREM 1. *Let R be a Noetherian domain. Then the following conditions are equivalent.*

1. t-*dim*$R = 1$.
2. *Each integrally closed t-linked overring of R is a subintersection of R'.*
3. *Each integrally closed t-linked overring of R is a Krull domain.*
4. *Each (proper if R is not a valuation domain) t-linked overring D of R is of t-dim$D = 1$.*
5. *Each t-linked valuation overring of R is a rank one DVR.*
6. *Each t-linked valuation overring of R is of dimension one.*

Proof. (1) ⇒ (2) Let D be an integrally closed t-linked overring of R and let Q be a maximal t-ideal of D. Since D is a t-linked overring of R, $Q \cap R$ is a prime t-ideal of R and so $\text{ht}(Q \cap R) = 1$. Since R' is integral over R and $Q \cap R = (Q = capR') \cap R$, $\text{ht}(Q \cap R') = 1$ and hence $R'_{Q \cap R'}$ is a rank one DVR (note that R' is a Krull domain [N, Theorem 33.10]). Since $R'_{Q \cap R'} \hookrightarrow D_Q \hookrightarrow K$, $R'_{Q \cap R'} = D_Q$ and hence ht$Q$=dim$D_Q = 1$. Thus t-dim$D = 1$ and $D = \cap_{Q \in X^1(D)} D_Q = \cap_{Q \in X^1(D)} R'_{Q \cap R'}$ is a subintersection of R'.

(2) ⇒ (3) Since R' is a Krull domain [[6], Theorem 33.10], a subintersection of R' is a Krull domain.

(3) ⇒ (1) Let P be a maximal t-ideal of R. Suppose that ht$P \geq 2$. Let V be a valuation overring of R with maximal ideal M such that $M \cap R = P$ and ht$P$=ht$M$ [G, Corollary 19.7]. Then V is a t-linked overring of R. For, if I is an ideal of R such that $I^{-1} = R$, then $I \nsubseteq P$ and hence $IV \nsubseteq M$ OR $IV = V$. since dim$V \geq 2$, V is not a Krull domain, a contradiction. Thus ht$P = 1$ and hence t-dim$R = 1$.

(1) ⇒ (4) Let Q be a maximal t-ideal of D and let $P := Q \cap R$. Then P is a prime t-ideal of R and so ht$P = 1$. Since $R_P \hookrightarrow D_Q \hookrightarrow K$ and R_P is a Noetherian domain, D_Q is an one-dimensional Noetherian domain [[5], Theorem 93]. Thus ht$Q$= dim$D_Q = 1$ and hence t-dim$D = 1$.

(3) $\Rightarrow$ (5) Let V a t-linked valuation overring of R. Since V is integrally closed, V is a Krull domain and so a rank one DVR.

(4) $\Rightarrow$ (6) and (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (1) Suppose that t-dim$R \geq 2$. Let $P_0 \subsetneq P_1$ be a chain of prime t-ideals of R. By [G, Corollary 19.7], there exists a valuation overring V of R with prime ideals $M_0 \subsetneq M_1$ such that $M_0 \cap R = P_0$ and $M_1 \cap R = M_1$. Since P_1 is a t-ideal of R, V is t-linked over R, a contradiction. Thus t-dim$R = 1$. □

In [[3], Theorem 9], Heinzer showed that if R is a Noetherian domain of dim$R \leq 2$, then each Krull overring of R is a Noetherian domain. Since an integrally closed Noetherian domain is a Krull domain, the following result is an immediate consequence of [[3], Theorem 9] and the above theorem.

Corollary 1. *Let R be a two-dimensional Noetherian domain. Then t-dim$R = 1$ if and only if each integrally closed t-linked overring of R is a Noetherian domain.*

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