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ONE-DIMENSIONAL PARABOLIC $p-$LAPLACIAN EQUATION

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Abstract. In this paper we establish some bounds for solutions of parabolic one dimensional p-Laplacian equation.

1. Introduction

We consider the Cauchy problem of the form

$$u_t = \left(|u_x|^{p-2}u_x\right)_x \quad \text{in} \quad S = \mathbb{R} \times [0, \infty) \tag{1.1}$$

where $p > 2$.

Equations like (1.1) were studied by many authors and arise in different physical situations, for the detail see [7]. An important quantity of the study of equation (1.1) is the local velocity of propagation $V(x,t)$, whose expression in terms of u can be obtained by writing the equation as a conservation law in the form

$$u_t + (uV)_x = 0.$$

In this way we get

$$V = -v_x|v_x|^{p-2},$$

where the nonlinear potential $v(x,t)$ is

$$v = \frac{p-1}{p-2}u^{\frac{p-2}{p-1}}. \tag{1.2}$$

and by direct computation v satisfies

$$v_t = (p-2)v|v_x|^{p-2}v_{xx} + |v_x|^p. \tag{1.3}$$

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In [7], it was shown that V satisfies

$$V_x \leq \frac{1}{2(p-1)t},$$

which can also be written as

$$(v_x|v_x|^{p-2})_x \geq -\frac{1}{2(p-1)t}. \tag{1.4}$$

Without loss of generality we may consider the case where u_0 vanishes on $\mathbb{R}^-$ and is a continuous positive function, at least, on an interval $(0, a)$ with $a > 0$. Let

$$P[u] = \{(x,t) \in S : u(x,t) > 0\}$$

be the positivity set of a solution u. Then $P[u]$ is bounded to the left in (x,t)-plane by the left interface curve $x = \zeta(t)$[7], where

$$\zeta(t) = \inf\{x \in \mathbb{R} : u(x,t) > 0\}.$$

Moreover there is a time $t^* \in [0, \infty)$, called the waiting time, such that $\zeta(t) = 0$ for $0 \leq t \leq t^*$ and $\zeta(t) < 0$ for $t > t^*$. It is shown [7] that t^* is finite(possibly zero) and $\zeta(t)$ is a nonincreasing C^1 function on (t^*, ∞).

For the interface of the porous medium equation

$$\begin{cases} u_t = \triangle(u^m) & \text{in} \quad \mathbb{R}^n \times [0, \infty), \\ u(x,0) = u_0 & \text{on} \quad \mathbb{R}^n \end{cases}$$

much more is known. D. G. Aronson and J. L. Vazquez [2] and independently K. Höllig and H. O. Kreiss [8] showed the interfaces are smooth after the waiting time. S. Angenent [1] showed that the interfaces are real analytic after the waiting time. In dimensions $n > 2$, L. A. Caffarelli and N. J. Wolanski [4] showed under some nondegeneracy conditions on the initial data, the interface can be described by a $C^{1,\alpha}$ function when $t > T$, for some $T > 0$. Very recently, P. Daskalopoulos and R. Hamilton [6] showed the interface is smooth when $0 < t < T$, for some $T > 0$.

On the other hand much less is known for the parabolic p-Laplacian equation. For dimensions $n > 2$, H. Choe and J. Kim [5] showed, under some nondegeneracy conditions on the initial data, the interface is Lipschitz continuous and one of the authors [9] improved this result, showing that, under the same hypotheses, the interface is a $C^{1,\alpha}$ surface after some time.

In [2], Aronson and Vazquez established C^∞ regularity of the interfaces by establishing the bounds for $v^{(k)}$ for $k \geq 2$, where $v = \frac{m}{m-1}u^{m-1}$

represents the pressure of the gas flow through a porous medium, while u represents the density. In this paper we establish bounds for v^k, $k = 2, 3$, near the interface after the waiting time, where v is the solution of (1.3).

2. The Upper and Lower Bounds for v_{xx}

Let $q = (x_0, t_0)$ be a point on the left interface, so that $x_0 = \zeta(t_0)$, $v(x, t_0) = 0$ for all $x \leq \zeta(t_0)$, and $v(x, t_0) > 0$ for all sufficiently small $x > \zeta(t_0)$. We assume the left interface is moving at q. Thus $t_0 > t^*$. We shall use the notation

$$R_{\delta,\eta} = R_{\delta,\eta}(t_0) = \{(x, t) \in \mathbb{R}^2 : \zeta(t) < x \leq \zeta(t) + \delta, t_0 - \eta \leq t \leq t_0 + \eta\}.$$

PROPOSITION 2.1. *Let q be the point as above. Then there exist positive constants C, δ and η depending only on p, q and u such that*

$$v_{xx} \geq C \quad \textit{in} \quad R_{\delta,\eta/2}.$$

Proof. From (1.4) we have, $v_{xx} \geq -\dfrac{1}{2(p-1)^2|v_x|^{p-2}t}$. But from Lemma 4.4 in [7] v_x is bounded away and above from zero near q where $u(x, t) > 0$. □

PROPOSITION 2.2. *Let $q = (x_0, t_0)$ be as before. Then there exist positive constants C_2, δ and η depending only on p, q and u such that*

$$v_{xx} \leq C_2 \quad \textit{in} \quad R_{\delta,\eta/2}.$$

Proof. From Theorem 2 and Lemma 4.4 in [7] we have

$$\zeta'(t_0) = -v_x|v_x|^{p-2} = -v_x^{p-1} = -a \tag{2.1}$$

and

$$v_t = |v_x|^p \tag{2.2}$$

on the moving part of the interface $\{x = \zeta(t), t > t^*\}$. Choose $\epsilon > 0$ such that

$$(p-1)a - 5p\epsilon \geq 4[(p-2)^2 + (p-1)^2](a + \epsilon)\epsilon. \tag{2.3}$$

Then by Theorem 2 in [7], there exists a $\delta = \delta(\epsilon) > 0$ and $\eta = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_{\delta,\eta} \subset P[u]$,

$$(a - \epsilon)^{\frac{1}{p-1}} < v_x < (a + \epsilon)^{\frac{1}{p-1}} \tag{2.4}$$

and

$$vv_{xx} \leq (a-\epsilon)^{\frac{2}{p-1}}\epsilon \tag{2.5}$$

in $R_{\delta,\eta}$. Then from (2.4) we have

$$(a-\epsilon)^{\frac{1}{p-1}}(x-\zeta) < v(x,t) < (a+\epsilon)^{\frac{1}{p-1}}(x-\zeta) \tag{2.6}$$

in $R_{\delta,\eta}$ and

$$-(a+\epsilon) < \zeta'(t) < -(a-\epsilon) \quad \text{in} \quad [t_1,t_2] \tag{2.7}$$

where $t_1 = t_0 - \eta$ and $t_2 = t_0 + \eta$. We set

$$\zeta^*(t) = \zeta(t_1) - b(t-t_1) \tag{2.8}$$

where $b = a + 2\epsilon$. Then clearly $\zeta(t) > \zeta^*(t)$ in $(t_1, t_2]$. On $P[u]$, $w \equiv v_{xx}$ satisfies

$$\begin{aligned} L(w) &= w_t - (p-2)v|v_x|^{p-2}w_{xx} - (3p-4)|v_x|^{p-2}v_x w_x \\ &\quad -[(p-2)^2 + 2(p-1)^2]|v_x|^{p-2}w^2 \\ &\quad -3(p-2)^2 v|v_x|^{p-4}v_x w w_x - (p-2)^2(p-3)v|v_x|^{p-4}w^3 \\ &= 0. \end{aligned}$$

We shall construct a barrier for w in $R_{\delta,\eta}$ of the form

$$\phi(x,t) \equiv \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)},$$

where α and β will be decided later.

By a direct computation we have

$$\begin{aligned} L(\phi) &= \frac{\alpha}{(x-\zeta)^2}\{\zeta' - (p-2)v|v_x|^{p-2}\frac{2}{x-\zeta} + (3p-4)|v_x|^{p-2}v_x\} \\ &\quad + \frac{\beta}{(x-\zeta^*)^2}\{\zeta^{*'} - (p-2)v|v_x|^{p-2}\frac{2}{x-\zeta^*} + (3p-4)|v_x|^{p-2}v_x\} \\ &\quad -[(p-2)^2 + 2(p-1)^2]|v_x|^{p-2}\phi^2 + \bar{G} \end{aligned}$$

where

$$\begin{aligned} \bar{G} &= -3(p-2)^2 v v_x |v_x|^{p-4}\phi\phi_x - (p-2)^2(p-3)v|v_x|^{p-4}\phi^3 \\ &= (p-2)^2 v|v_x|^{p-4} \times \\ &\quad \phi\left(3v_x[\frac{\alpha}{(x-\zeta)^2} + \frac{\beta}{(x-\zeta^*)^2}] - (p-3)[\frac{\alpha}{x-\zeta} + \frac{\beta}{x-\zeta^*}]^2\right). \end{aligned}$$

If we choose α and β satisfying

$$v_x \geq |p-3| \max(\alpha, \beta),$$

then $\bar{G} \geq 0$ in $R_{\delta,\eta}$. Now set $\bar{A} = \frac{\alpha}{(x-\zeta)^2}$ and $\bar{B} = \frac{\beta}{(x-\zeta^*)^2}$. Then we have

$$\begin{aligned} L(\phi) \geq \bar{A}\Big\{&\zeta' + |v_x|^{p-2}\{-(p-2)v\frac{2}{x-\zeta} + (3p-4)v_x \\ &- 2[(p-2)^2 + 2(p-1)^2]\alpha\}\Big\} \\ + \bar{B}\Big\{&\zeta^{*'} + |v_x|^{p-2}\{-(p-2)v\frac{2}{x-\zeta^*} + (3p-4)v_x \\ &- 2[(p-2)^2 + 2(p-1)^2]\beta\}\Big\} \\ \geq \bar{A}&\left\{(p-1)a - (5p-7)\epsilon - 2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\alpha\right\} \\ + \bar{B}&\left\{(p-1)a - (5p-6)\epsilon - 2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\beta\right\}. \end{aligned}$$

Set

$$0 < \alpha \leq \frac{(p-1)a - (5p-7)\epsilon}{2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}} = \alpha_0$$

and

$$\beta = \frac{(p-1)a - (5p-6\epsilon)}{2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}}. \tag{2.9}$$

Then from (2.3) $\beta > 0$ and $L(\phi) \geq 0$ in $R_{\delta,\eta}$ for all $\alpha \in (0, \alpha_0]$ and β.

Let us now compare w and ϕ on the parabolic boundary of $R_{\delta,\eta}$. In view of (2.5) and (2.6) we have

$$v_{xx} \leq \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta} \quad \text{in} \quad R_{\delta,\eta}$$

and in particular

$$v_{xx}(\zeta(t)+\delta, t) \leq \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{\delta} \quad \text{in} \quad [t_1, t_2].$$

By the mean value theorem and (2.7) we have for some $\tau \in (t_1, t_2)$

$$\begin{aligned} \zeta(t) + \delta - \zeta^*(t) &= \delta + (a+2\epsilon)(t-t_1) + \zeta'(\tau)(t-t_1) \\ &\leq \delta + 3\epsilon(t-t_1) \leq \delta + 6\epsilon\eta. \end{aligned}$$

Now set

$$\eta \equiv \min\{\eta(\epsilon), \delta(\epsilon)/6\epsilon\}.$$

Since ϵ satisfies (2.3) and β is given by (2.9) it follows that

$$\phi(\zeta+\delta,t) \geq \frac{\beta}{2\delta} \geq \frac{(p-1)a-(5p-6\epsilon)}{4[(p-2)^2+2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\delta}$$
$$\geq \frac{(a+\epsilon)^{\frac{1}{p-1}}}{\delta}\epsilon \geq v_{xx} \quad \text{on} \quad [t_1,t_2]$$

Moreover from (3.5) and (2.9)

$$\phi(x,t_1) \geq \frac{\beta}{x-\zeta(t_1)} > \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta(t_1)} > v_{xx}(x,t_1) \quad \text{on} \quad (\zeta(t_1),\zeta(t_1)+\delta].$$

Let $\Gamma = \{(x,t) \in \mathbb{R}^2 : x = \zeta(t), t_1 \leq t \leq t_2\}$. Clearly Γ is a compact subset of $\mathbb{R}^2$. Fix $\alpha \in (0,\alpha_0)$. For each point $s \in \Gamma$ there is an open ball B_s centered at s such that

$$(vv_{xx})(x,t) \leq \alpha(a-\epsilon)^{\frac{1}{p-1}} \quad \text{in} \quad B_s \cap P[u].$$

In view of (2.6) we have

$$\phi(x,t) \geq \frac{\alpha}{x-\zeta} \geq v_{xx}(x,t) \quad \text{in} \quad B_s \cap P[u].$$

Since Γ can be covered by a finite number of these balls it follows that there is a $\gamma = \gamma(\alpha) \in (0,\delta)$ such that

$$\phi(x,t) \geq w(x,t) \quad \text{in} \quad R_{\delta,\eta}.$$

Thus for every $\alpha \in (0,\alpha_0)$, ϕ is a barrier for w in $R_{\delta,\eta}$. By the comparison principle for parabolic equations [10] we conclude that

$$v_{xx}(x,t) \leq \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta},$$

where β is given by (2.9) and $\alpha \in (0,\alpha_0)$ is arbitrary. Now let $\alpha \downarrow 0$ to obtain

$$v_{xx}(x,t) \leq \frac{\beta}{x-\zeta^*} \leq \frac{2\beta}{\epsilon\eta} \quad \text{in} \quad \mathbb{R}. \qquad \square$$

3. Bounds for $\left(\frac{\partial}{\partial x}\right)^3 v$

In this section we find the estimates of the derivatives of the form

$$v^{(3)} \equiv \left(\frac{\partial}{\partial x}\right)^3 v.$$

By a direct computation we have,

$$\begin{aligned}L_3(v^{(3)}) &= v_t^{(3)} - (p-2)vv_x^{p-2}v_{xx}^{(3)} - (A+B)v_x^{(3)} - Cv^{(3)} - D(v^{(3)})^2 \\ &\quad - Ev_x^{p-3}v_{xx}^3 - (p-2)^2(p-3)(p-4)vv_x^{p-5}v_{xx}^4 = 0\end{aligned} \tag{3.1}$$

where

$$A = (p-2)v_x^{p-1} + (p-2)^2 vv_x^{p-3}v_{xx},$$

$$B = (3p-4)v_x^{p-1} + 3(p-2)^2 vv_x^{p-3}v_{xx},$$

$$\begin{aligned}C = v_{xx}v_x^{p-2}\{&(3p-4)(p-1) \\ &+ 2[(p-2)^2 + 2(p-1)^2] + 6(p-2)^2(p-3)vv_x^{-2}v_{xx} + 3(p-2)^2\},\end{aligned}$$

$$D = 3(p-2)^2 vv_x^{p-3},$$

$$E = [(p-2)^2 + 2(p-1)^2](p-2) + (p-2)^2(p-3).$$

Suppose that $q = (x_0, t_0)$ is a point on the left interface for which (2.1) holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0,\eta_0}(t_0) \subset P[u]$ and (2.5) holds. Thus we also have (2.6) and (2.7) in R_0. Then by rescaling and interior estimate we have

Proposition 3.1. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0, \delta_0)$, and $\eta \in (0, \eta_0)$ depending only on p,q and C_2 such that*

$$|v^{(3)}(x,t)| \le \frac{K}{x - \zeta(t)} \quad \text{in} \quad R_{\delta,\eta}.$$

Proof. Set

$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\},$$

$$\eta = \eta_0 - \frac{\delta}{4s},$$

and define

$$R(\overline{x}, \overline{t}) \equiv \left\{(x,t) \in \mathbb{R}^2 : |x - \overline{x}| < \frac{\lambda}{2}, \overline{t} - \frac{\lambda}{4s} < t \le \overline{t}\right\}$$

for $(\overline{x}, \overline{t}) \in R_{\delta,\eta}$, where $s = a + \epsilon$ and $\lambda = \overline{x} - \zeta(\overline{t})$. Then $(\overline{x}, \overline{t}) \in R_{\delta,\eta}$ implies that $R(\overline{x}, \overline{t}) \subset R_0$. Since $\delta_0 \ge \frac{3\delta}{2}$, $\lambda < \delta$ and ζ is nonincreasing, we have

$$t_0 - \eta_0 = t_0 - \eta - \frac{\lambda}{4s} < t < t_0 + \eta < t_0 + \eta_0$$

and

$$\overline{x} - \frac{\lambda}{2} = \overline{x} - \frac{\overline{x} + \zeta(\overline{t})}{2} = \frac{\overline{x} + \zeta(\overline{t})}{2} > \zeta(t_0 + \eta_0)$$

$$\zeta(t_0-\eta)+\delta+\frac{\lambda}{2}<\zeta(t_0-\eta_0).$$

Also observe that for each $(\overline{x},\overline{t})\in R_{\delta,\eta}$, $R(\overline{x},\overline{t})$ lies to the right of the line $x=\zeta(\overline{t})+s(\overline{t}-t)$. Next set $x=\lambda\xi+\overline{x}$ and $t=\lambda\tau+\overline{t}$. The function

$$W(\xi,\tau)\equiv v_{xx}(\lambda\xi+\overline{x},\lambda\tau+\overline{t})=v_{xx}(x,t)$$

satisfies the equation

$$\begin{aligned} W_\tau &= \left\{(p-2)\frac{v}{\lambda}v_x^{p-2}W_\xi+(3p-4)v_x^{p-1}W\right\}_\xi \\ &+[2(p-2)^2vv_x^{p-3}v_{xx}-(p-2)v_x^{p-1}]W_\xi \\ &+\lambda[(p-2)^2(p-3)vv_x^{p-4}(v_{xx})^3-(p-2)v_x^{p-2}(v_{xx})^2] \end{aligned} \tag{3.2}$$

in the region

$$B\equiv\left\{(\xi,\tau)\in\mathbb{R}^2:|\xi|\le\frac{1}{2},-\frac{1}{4s}<\tau\le 0\right\},$$

and $|W|\le C_2$ in B. In view of (2.6) and (2.7)

$$(a-\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}\le\frac{v(x,t)}{\lambda}\le(a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}$$

and

$$\zeta(\overline{t})\le\zeta(t)\le\zeta(\overline{t})+s(\overline{t}-t)\le\zeta(\overline{t})+\frac{\lambda}{4}.$$

Therefore

$$\frac{\lambda}{4}=\overline{x}-\frac{\lambda}{2}-\zeta(\overline{t})-\frac{\lambda}{4}\le x-\zeta(t)\le\overline{x}+\frac{\lambda}{2}-\zeta(\overline{t})=\frac{3\lambda}{2}$$

which implies

$$\frac{(a-\epsilon)^{\frac{1}{p-1}}}{4}\le\frac{v}{\lambda}\le\frac{3(a+\epsilon)^{\frac{1}{p-1}}}{2}.$$

Hence by (2.4) equation (3.2) is uniformly parabolic in B. Moreover, it follows from Proposition 2.2 that W satisfies all of the hypotheses of Theorem 5.3.1 of [10]. Thus we conclude that there exists a constant $K=K(a,p,C_2)>0$ such that

$$\left|\frac{\partial}{\partial\xi}W(0,0)\right|\le K;$$

that is,

$$|v^{(3)}(\overline{x},\overline{t})|\le\frac{K}{\lambda}.$$

Since $(\overline{x},\overline{t})\in R_{\delta,\eta}$ is arbitrary, this proves the proposition. □

We now turn to the barrier construction. If $\gamma \in (0,\delta)$ we will use the notation

$$R^{\gamma}_{\delta,\eta} = R^{\gamma}_{\delta,\eta}(t_0) \equiv \{(x,t) \in \mathbb{R}^2 : \zeta(t)+\gamma \le x \le \zeta(t)+\delta, t_0-\eta \le t \le t_0+\eta\}.$$

PROPOSITION 3.2. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2 with*

$$0 < \delta_1 < \frac{(p-1)a^{\frac{1}{p-1}}}{12(p-2)^2 K}. \tag{3.3}$$

For $(x,t) \in R^{\gamma}_{\delta_1,\eta_1}$, let

$$\phi_\gamma(x,t) \equiv \frac{\alpha}{x-\zeta(t)-\gamma/3} + \frac{\beta}{x-\zeta^*(t)} \tag{3.4}$$

where ζ^* is given by (2.8), and α and β are positive constant less than $K/2$. Then there exist $\delta \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ depending only on a, p and C_2 such that

$$L_3(\phi_\gamma) \ge 0 \quad in \quad R^{\gamma}_{\delta,\eta}$$

for all $\gamma \in (0,\delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{(p-1)a}{13p-23}. \tag{3.5}$$

There exist $\delta_2 \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ such that (2.4), (2.6) and (2.7) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0,\delta_2)$. For $(x,t) \in R^{\gamma}_{\delta_2,\eta}$, we have

$$\begin{aligned} L_3(\phi_3) = {} & \frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{\zeta' - \frac{2(p-2)vv_x^{p-2}}{x-\zeta-\gamma/3} + A + B\right\} \\ & + \frac{\alpha}{(x-\zeta^*)^2}\left\{\zeta^{*\prime} - \frac{2(p-2)vv_x^{p-2}}{x-\zeta^*} + A + B\right\} \\ & - C\phi_3 - D(\phi_3)^2 - Ev_x^{p-3}v_{xx}^3 - (p-2)^2(p-3)(p-4)vv_x^{p-5}v_{xx}^4 \end{aligned}$$

where A, B, C, D and E are as before.

From (2.6), together with the fact that $x-\zeta^* \ge x-\zeta-\gamma/3$ we have

$$\begin{aligned} \frac{v}{x-\zeta^*} \le \frac{v}{x-\zeta-\gamma/3} & \le (a+\epsilon)^{\frac{1}{p-1}} \frac{x-\zeta}{x-\zeta-\gamma/3} \\ & \le (a+\epsilon)^{\frac{1}{p-1}} \frac{\gamma}{\gamma-\gamma/3} = \frac{3}{2}(a+\epsilon)^{\frac{1}{p-1}}. \end{aligned}$$

From (3.3), we have

$$D\alpha, D\beta < \frac{DK}{2} < DK \le \frac{(p-1)a}{4} + \frac{(p-1)\epsilon}{4}. \tag{3.6}$$

Then since $|C|$ is bounded and from (2.4) and (2.6), we have

$$\begin{aligned} L_3(\phi_3) \ge & \frac{\alpha}{Y^2}\Big\{(p-1)a - (7p-11)\epsilon - |C|Y - 2D\alpha - \overline{E}\frac{Y^2}{\alpha}\Big\} \\ & + \frac{\beta}{(x-\zeta^*)^2}\Big\{(p-1)a - (7p-10)\epsilon - |C|(x-\zeta^*) \\ & \quad - 2D\beta - \overline{E}\frac{(x-\zeta^*)^2}{\beta}\Big\} \\ \ge & \frac{\alpha}{Y^2}\Big\{\frac{(p-1)a}{2} - \frac{13p-23}{2}\epsilon - \delta_2(|C| - \overline{E}\frac{Y}{\alpha})\Big\} \\ & + \frac{\beta}{(x-\zeta^*)^2}\Big\{\frac{(p-1)a}{2} - \frac{13p-21}{2}\epsilon - \delta_2(|C| - \overline{E}\frac{x-\zeta^*}{\beta})\Big\} \end{aligned}$$

where $Y = x - \zeta - \gamma/3$ and $\overline{E} = |E| v_x^{p-3} v_{xx}^3$. Since ϵ satisfies (3.5) we can choose $\delta = \delta_2(\epsilon, p, a, C_2) > 0$ so small that $L_3(\phi_3) \ge 0$ in $R^{\gamma}_{\delta,\eta}$. □

Remark 3.1. *From (3.6) the Proposition 3.2 will be true for any* $\alpha, \beta \in (0, K)$.

Proposition 3.3. *(Barrier Transformation). Let* δ *and* η *be as in Proposition 3.2 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{3.7}$$

where ϵ *is as in Proposition 3.2. Suppose that for some nonnegative constant* β

$$v^{(3)}(x,t) \le \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta}. \tag{3.8}$$

Then $v^{(3)}$ *also satisfies*

$$v^{(3)}(x,t) \le \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta+2\alpha/3}{x-\zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta}. \tag{3.9}$$

Proof. By Remark 3.1, for any $\gamma \in (0,\delta)$ since $\beta + 2\alpha/3 \le K$ the function

$$\phi_3(x,t) = \frac{2\alpha/3}{x-\zeta-\gamma/3} + \frac{\beta+2\alpha/3}{x-\zeta^*}$$

satisfies $L_3(\phi_3) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. On the other hand, on the parabolic boundary of $R^{\gamma}_{\delta,\eta}$ we have $\phi_3 \geq v^{(3)}$. In fact, for $t = t_1$ and $\zeta_1 + \gamma \leq x \leq \zeta_1 + \delta$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_3(x, t_1) = \frac{2\alpha}{x - \zeta_1 - \gamma/3} + \frac{\beta + 2\alpha/3}{x - \zeta_1} > \frac{4\alpha/3}{x - \zeta_1} + \frac{\beta}{x - \zeta_1} > v^{(3)}(x, t_1)$$

while for $x = \zeta + \delta$ and $t_1 \leq t \leq t_2$ we get, in view of (3.7),

$$\begin{aligned} \phi_3(\zeta + \delta, t) &\geq \frac{2\alpha/3}{\delta - \gamma/3} + \frac{\beta}{\zeta + \delta - \zeta^*} + \frac{2\alpha/3}{\delta + 6\epsilon\eta} \\ &\geq \frac{2\alpha/3}{\delta} + \frac{\delta}{\zeta + \delta - \zeta^*} + \frac{\alpha/3}{\delta} \geq v^{(3)}(\zeta + \delta, t). \end{aligned}$$

Finally, for $x = \zeta + \gamma$, $t_1 \leq t \leq t_2$ we have

$$\phi_3(\zeta + \delta, t) = \frac{2\alpha/3}{\gamma - \gamma/3} + \frac{\beta + 2\alpha/3}{\zeta + \gamma - \zeta^*} \geq \frac{\alpha}{\gamma} + \frac{\beta}{\zeta + \gamma - \zeta^*} \geq v^{(3)}(\zeta + \gamma, t).$$

By the comparison principle we get

$$\phi_3 \geq v^{(3)} \quad \text{in} \quad R^{\gamma}_{\delta.\eta}$$

for any $\gamma \in (0, \delta)$, and (3.9) follows by letting $\gamma \downarrow 0$. □

Proposition 3.4. *Let $q = (x_0, t_0)$ be a point on the interface for which (2.1) holds. Then there exist constants C_3, δ and η depending only on p, q and u such that*

$$\left| \left(\frac{\partial}{\partial x} \right)^3 v \right| \leq C_3 \quad \textit{in} \quad R_{\delta,\eta/2}.$$

Proof. By Proposition 3.1 we have, by letting $\alpha = 0$,

$$v^{(3)}(x, t) \leq \frac{\beta}{x - \zeta^*} \leq \frac{2\beta}{\epsilon\eta} \quad \text{in} \quad R_{\delta,\eta/2}.$$

Even though the equation (3.1) is not linear for $v^{(3)}$, a lower bound can be obtained in a similar way. □

References

[1] S. Angenent, Analyticity of the interface of the porous media equation after the waiting time, *Proc. Amer. Math. Soc.* **102** (1988), 329-336.

[2] D. G. Aronson and J. L. Vazquez, Eventual C^{∞}-regularity and concavity for flows in one-dimensional porous media, *Arch. Rational Mech. Anal.* **99** (1987),no.4, 329-348.

[3] L. A. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, *Ind. Univ. Math. J.* **29** (1980), 361-381.

[4] L. A. caffarelli and N. J. Wolanski, $C^{1,\alpha}$ regularity of the free boundary for the N-dimensional porous medium equation, *Comm. Pure Appl. Math.* **43** (1990), 885-902.

[5] H. Choe and J. Kim, Regularity for the interfaces of evolutionary p-Laplacian functions, *Siam J. Math. Anal.* **26** (**4**) (1995), 791-819.

[6] P. Daskalopoulos and R. Hamilton, Regularity of the free boundary for the porous medium equation, *J. A.M.S.* **11**(**4**) (1998), 899-965.

[7] J. R. Esteban and J. L. Vazquez, Homogeneous diffusion in R with power-like nonlinear diffusivity, *Arch. Rational Mech. Anal.* **103**(1988) 39-80.

[8] K. Höllig and H. O. Kreiss, C^{∞}-Regularity for the porous medium equation, *Math. Z.* **192** (1986), 217-224.

[9] Y. Ko, $C^{1,\alpha}$ regularity of interfaces for solutopns of the parabolic p-Laplacian equation, *Comm. in P. D. E.*, to appear.

[10] O. A. Ladyzhenskaya, N.A. Solonnikov and N.N. Uraltzeva, Linear and quasi-linear equations of parabolic type, *Trans. Math. Monographs,* **23**, Amer. Math. Soc., Providence, R. I., 1968.

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