Kangweon-Kyungki Math. Jour. **7** (1999), No. 1, pp. 103–110

GROUP ACTIONS ON KAC ALGEBRAS

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Abstract. For a group action α on a Kac algebra $\mathbb{K}$ with the crossed product Kac algebra $\mathbb{K} \rtimes_\alpha G$, we will show that $\pi_\alpha(\mathbb{K})$ is a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$. We will also investigate the intrinsic group $G(\mathbb{K})$ of $\mathbb{K}$ and get a group action β on a symmetric Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ with the crossed product sub-Kac algebra $\mathbb{K}_s(G(\mathbb{K})) \rtimes_\beta G$ of $\mathbb{K} \rtimes_\alpha G$.

1. Introduction

Kac algebra theory is one of the most important topics in recent operator algebra theory and much effort has been made to develope it ([E], [EN], [ES2], [KP], [NT], [Sc], [TT], [W]). As is well known, typical examples of von Neumann algebra come from group actions. So it is interesting to consider group actions on Kac algebras ([DeC1], [ES1], [N], [Y2]). Of course, outer actions on von Neumann algebras are very useful. But the action in this paper is different from outer actions and gives examples of Kac algebras.

Furthermore, since the intrinsic group of a Kac algebra consists of "group-like" elements of the given Kac algebra, it can be considered as a natural kind of invariant attached to each Kac algebra. So to study the intrinsic group is one of the important things in the theory of Kac algebras ([DeC2], [Y1]).

For a locally compact group G and a G-action α on a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$, we have the crossed product Kac algebra $\mathbb{K} \rtimes_\alpha G$ and an inclusion $\pi_\alpha(M) \subset M \rtimes_\alpha G$ of von Neumann alebras can be made into an inclusion $\pi_\alpha(\mathbb{K}) \subset \mathbb{K} \rtimes_\alpha G$ of Kac algebras. Moreover, we have that $\pi_\alpha(G(\mathbb{K}))$ is a subgroup of the intrinsic group $G(\mathbb{K} \rtimes_\alpha G)$ and there exists a G-action β on a maximal abelian sub-Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ of

Received January 2, 1999.
1991 Mathematics Subject Classification: Primary 46L40, Secondary 46L55.
Key words and phrases: Kac algebra, a G-action on a Kac algebra, crossed product Kac algebra.

$\mathbb{K}$ with the crossed product sub-Kac algebra $\mathbb{K}_s(G(\mathbb{K})) \rtimes_\beta G$ of $\mathbb{K} \rtimes_\alpha G$. Finally, we will give some properties and examples of group actions on Kac algebras.

2. Preliminaries

In this section, we introduce notations and briefly review fundamental results which will be necessary for our discussion.

In order to fix notations, we first describe the notion of Kac algebras defined by [ES2]. For the general theory of Kac algebras, we refer to Enock and Schwartz [ES2].

A triplet $(M,\ \Gamma,\ \kappa)$ consisting of a von Neumann algebra M, an isomorphism $\Gamma : M \to M \otimes M$ satisfying coproduct condition $(\Gamma \otimes i) \circ \Gamma = (i \otimes \Gamma) \circ \Gamma$ and an involutive antiisomorphism $\kappa : M \to M$ with $\sigma \circ \Gamma \circ \kappa = (\kappa \otimes \kappa) \circ \Gamma$ is called an involutive Hopf-von Neumann algebra $\mathbb{H}$, where σ is a flip.

DEFINITION 2.1. An involutive Hopf-von Neumann algebra $\mathbb{H} = (M,\ \Gamma,\ \kappa)$ with a Haar weight φ is called a Kac algebra and denoted by $\mathbb{K} = (M,\ \Gamma,\ \kappa,\ \varphi)$.

In this case, M is represented standardly on the Hilbert space $\mathcal{H}_\varphi$ obtained from φ.

For two Kac algebras $\mathbb{K}_i = (M_i, \Gamma_i, \kappa_i, \varphi_i), (i = 1, 2)$, we shall say that $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic (i.e. $\mathbb{K}_1 \cong \mathbb{K}_2$) if there exist an $\mathbb{H}$-isomorphism $u : (M_1, \Gamma_1, \kappa_1) \to (M_2, \Gamma_2, \kappa_2)$ and $k > 0$ such that $\varphi_2 \circ u = k\varphi_1$. Note that an $\mathbb{H}$-isomorphism u is a unital normal isomorphism from M_1 to M_2 such that $\Gamma_2 u = (u \otimes u)\Gamma_1$ and $\kappa_2 u = u\kappa_1$.

For every Kac algebra $\mathbb{K} = (M,\ \Gamma,\ \kappa,\ \varphi)$, there canonically exists its dual Kac algebra $\hat{\mathbb{K}} = (\hat{M},\ \hat{\Gamma},\ \hat{\kappa},\ \hat{\varphi})$ and whose underlying von Neumann algebra $\hat{M}$ is also represented standardly on the same space $\mathcal{H}_\varphi$. Note that the dual Kac algebra of $\hat{\mathbb{K}}$ is isomorphic to $\mathbb{K}$.

Now we consider a von Neumann subalgebra M_0 of M such that:

(1) $\Gamma(M_0) \subset M_0 \otimes M_0$.
(2) $\kappa(M_0) = M_0$.
(3) the restriction $\varphi_0 = \varphi|_{M_0^+}$ is a semi-finite weight.

We shall denote by $\mathbb{K}_0$, the quadruple $(M_0,\ \Gamma_0,\ \kappa_0,\ \varphi_0)$ where Γ_0 and κ_0 are respectively the restriction of Γ and κ on M_0. Then it is well known that $\mathbb{K}_0$ is a Kac algebra, called a sub-Kac algebra of $\mathbb{K}$.

The intrinsic group $G(\mathbb{K})$ of a Kac algebra $\mathbb{K}$ consists of all invertible elements $x \in M$ such that $\Gamma(x) = x \otimes x$. We know that $G(\mathbb{K})$ is a closed subgroup of the unitary group of M, when equipped with the weak topology.

The basic examples of Kac algebras that we shall need, are ones associated to locally compact groups. From now on, G denotes a locally compact group G with a left Haar measure ds. The left regular unitary representation λ of G on $L^2(G)$ is defined by $(\lambda(g)\xi)(h) = \xi(g^{-1}h), (\xi \in L^2(G), g, h \in G)$. Then the group von Neumann algebra $L(G)$ is defined by $\{\lambda(g)|g \in G\}''$.

As is well known, the crossed product $M \rtimes_\alpha G$ of M by G relative to a G-action α is the von Neumann algebra $\{\pi_\alpha(M) \cup \{\rho(g)\}_{g\in G}\}''$, where ρ is the right regular representation of G on $L^2(G)$ and a normal injective homomorphism $\pi_\alpha : M \to M \otimes L^\infty(G)$ is defined by $(\pi_\alpha(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g), (\xi \in L^2(G, \mathcal{H}), g \in G)$ with $\rho(g)\pi_\alpha(x)\rho(g)^* = \pi_\alpha(\alpha_g(x))$.

We associate two concrete Kac algebras acting on $L^2(G)$. One is the abelian Kac algebra $\mathbb{K}_a(G) = (L^\infty(G), \Gamma_a, \kappa_a, \varphi_a)$, where

$$\Gamma_a(f)(s,t) = f(st), \quad \kappa_a(f)(s) = f(s^{-1}), \quad \varphi_a(f) = \int_G f(s)ds.$$

The other is the symmetric Kac algebra $\mathbb{K}_s(G) = (L(G), \Gamma_s, \kappa_s, \varphi_s)$, where

$$\Gamma_s(\lambda(s)) = \lambda(s) \otimes \lambda(s), \quad \kappa_s(\lambda(s)) = \lambda(s^{-1}),$$

the weight φ_s is the so-called Plancherel weight of G. Note that $\mathbb{K}_a(G)$ and $\mathbb{K}_s(G)$ are dual to each other.

The group of all continuous characters on G, denoted by $\chi(G)$ is a topological group with the topology of compact convergence. The intrinsic group $G(\mathbb{K}_a(G))$ of a Kac algebra $\mathbb{K}_a(G)$ can be identified with $\chi(G)$ and $G(\mathbb{K}_s(G)) \cong G$ (see [DeC2]).

Now consider group actions on Kac algebras. Given a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$ and G-action α on M, we recall the definition of a G-action on a Kac algebra $\mathbb{K}$ (see Definition 2.7 in [DeC1]).

DEFINITION 2.2. Let $\mathbb{K} = (M,\ \Gamma,\ \kappa,\ \varphi)$ be a Kac algebra and G a locally compact group. A σ-weakly continuous action α of G on M is said to be a G-action on $\mathbb{K}$ if it satisfies the following:

$$\Gamma \circ \alpha_g = (\alpha_g \otimes \alpha_g) \circ \Gamma, \quad \kappa \circ \alpha_g = \alpha_g \circ \kappa, \quad (g \in G).$$

It was shown in [DeC1] that for a G-action α on $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$, $M \rtimes_\alpha G$ can be made into a Kac algebra $\mathbb{K} \rtimes_\alpha G = (M \rtimes_\alpha G, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$, called crossed product Kac algebra by a locally compact group, satisfying $\tilde{\Gamma}(\pi_\alpha(x)) = (\pi_\alpha \otimes \pi_\alpha)\Gamma(x)$ and $\tilde{\kappa}(\pi_\alpha(x)) = \pi_\alpha \circ \kappa(x), (x \in M)$.

3. Group Actions on Kac algebras and crossed product Kac algebras

In this section, we consider a G-action α on a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$ with the crossed product Kac algebra $\mathbb{K} \rtimes_\alpha G = (M \rtimes_\alpha G, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$. Note that we get an inclusion $\pi_\alpha(M) \subset M \rtimes_\alpha G$ of von Neumann algebras.

The following proposition tells that we can get an inclusion of Kac algebras $\mathbb{K} \cong \pi_\alpha(\mathbb{K}) \subset \mathbb{K} \rtimes_\alpha G$. In order to do this, we need that for two Kac algebras $\mathbb{K}_i = (M_i, \Gamma_i, \kappa_i, \varphi_i), (i = 1, 2)$, if there exists a unital normal isomorphism $u : M_1 \to M_2$ such that $\Gamma_2 u = (u \otimes u)\Gamma_1$ then by 5.5.6 in [ES2], u is an $\mathbb{H}$-isomorphism from $(M_1, \Gamma_1, \kappa_1)$ to $(M_2, \Gamma_2, \kappa_2)$. From 2.7.9 in [ES2] we get that there exists $k > 0$ such that $\varphi_2 \circ u = k\varphi_1$, and so two Kac algebras $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic.

PROPOSITION 3.1. *Let α be a G-action on a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$. Then $\pi_\alpha(M)$ can be made into a sub-Kac algebra $\pi_\alpha(\mathbb{K})$ of $\mathbb{K} \rtimes_\alpha G$.*

Proof. From the properties of $\tilde{\Gamma}$ and $\tilde{\kappa}$, we see that $\tilde{\Gamma}(\pi_\alpha(M)) \subset \pi_\alpha(M) \otimes \pi_\alpha(M)$ and $\tilde{\kappa}(\pi_\alpha(M)) = \pi_\alpha(M)$. It is well known that the restriction of a Haar weight $\tilde{\varphi}$ on $\pi_\alpha(M)$ is a semi-finite weight. Thus, $\pi_\alpha(\mathbb{K})$, whose underlying von Neumann algebra is $\pi_\alpha(M)$, becomes a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$.

Note that we can identify M and $\pi_\alpha(M)$, which gives $\mathbb{K} \cong \pi_\alpha(\mathbb{K})$, as in the above statement. □

We give here a nice behavior of automorphisms $\alpha_g, g \in G$ of M, where α is a G-action on $\mathbb{K}$.

LEMMA 3.2. *Let α be a G-action on a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$. Then for any $g \in G$, we have $\alpha_g(G(\mathbb{K})) = G(\mathbb{K})$, where $G(\mathbb{K})$ is the intrinsic group of $\mathbb{K}$.*

Proof. For any $g \in G$ and $x \in G(\mathbb{K})$, since α is a G-action on $\mathbb{K}$, we have

$$\Gamma(\alpha_g(x)) = (\alpha_g \otimes \alpha_g)(\Gamma(x)) = (\alpha_g \otimes \alpha_g)(x \otimes x) = \alpha_g(x) \otimes \alpha_g(x),$$

which implies $\alpha_g(x) \in G(\mathbb{K})$ and so $\alpha_g(G(\mathbb{K})) \subset G(\mathbb{K})$.

On the other hand, the fact of $\alpha_g^{-1} = \alpha_{g^{-1}}, (g \in G)$ gives $G(\mathbb{K}) \subset \alpha_g(G(\mathbb{K}))$, which completes the proof. □

LEMMA 3.3. *Let $\mathbb{K}_0 = (M_0, \Gamma_0, \kappa_0, \varphi_0)$ be a sub-Kac algebra of $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$ and α a G-action on $\mathbb{K}$. If the restriction β_g of $\alpha_g, (g \in G)$ on M_0 is an automorphism of M_0 then β is a G-action on $\mathbb{K}_0$ such that the crossed product Kac algebra $\mathbb{K}_0 \rtimes_\beta G$ whose underlying von Neumann algebra is $M_0 \rtimes_\beta G$, is a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$.*

Proof. Since $\beta_g = \alpha_g|_{M_0}, (g \in G)$ is an automorphism of M_0 and α is a G-action on M, clearly, β is a G-action on M_0.

For any $g \in G$ and $x \in M_0$, the facts of $\Gamma(x) = \Gamma_0(x) \in M_0 \otimes M_0$, $\kappa(x) = \kappa_0(x) \in M_0$ and $\beta_g(x) = \alpha_g(x)$ give that

$$\Gamma_0(\beta_g(x)) = \Gamma(\alpha_g(x)) = (\alpha_g \otimes \alpha_g)(\Gamma(x)) = (\beta_g \otimes \beta_g)(\Gamma_0(x))$$

and

$$\kappa_0(\beta_g(x)) = \kappa(\alpha_g(x)) = \alpha_g(\kappa(x)) = \beta_g(\kappa_0(x)),$$

which implies that β is a G-action on a Kac algebra $\mathbb{K}_0$.

It is straightforward to show that $\mathbb{K}_0 \rtimes_\beta G$ is a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$. □

Now we are ready to prove that there exists a G-action on the symmetric Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ with the crossed product sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$.

THEOREM 3.4. *For a G-action α on a Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$, we have the following:*

(1) *$\pi_\alpha(G(\mathbb{K}))$ is a subgroup of $G(\mathbb{K} \rtimes_\alpha G)$.*
(2) *There exists a G-action β on a maximal abelian sub-Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ of $\mathbb{K}$ and the crossed product Kac algebra $\mathbb{K}_s(G(\mathbb{K})) \rtimes_\beta G$ is a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$.*

Proof.

(1) For any x in $G(\mathbb{K})$, we have $\tilde{\Gamma}(\pi_\alpha(x)) = (\pi_\alpha \otimes \pi_\alpha)\Gamma(x) = (\pi_\alpha \otimes \pi_\alpha)(x \otimes x) = \pi_\alpha(x) \otimes \pi_\alpha(x)$, which implies $\pi_\alpha(G(\mathbb{K})) \subset G(\mathbb{K} \rtimes_\alpha G)$. Since $G(\mathbb{K})$ is a subgroup of the unitary group of M and π_α is a homomorphism, $\pi_\alpha(G(\mathbb{K}))$ is a subgroup of $G(\mathbb{K} \rtimes_\alpha G)$.

(2) By Lemma 3.2, $\alpha_g(G(\mathbb{K})) = G(\mathbb{K}), g \in G$ and $\mathbb{K}_s(G(\mathbb{K}))$ is a maximal abelian sub-Kac algebra of $\mathbb{K}$ (see [ES2]). For any $g \in G$ and $x \in G(\mathbb{K}))$, if we define β_g by $\beta_g(\lambda(x)) = \lambda(\alpha_g(x))$ then it is straightforward to show that β is a G-action on $L(G(\mathbb{K}))$. Thus by Lemma 3.3, β is a G-action on a Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ and $\mathbb{K}_s(G(\mathbb{K})) \rtimes_\beta G$ is a sub-Kac algebra of $\mathbb{K} \rtimes_\alpha G$. □

Among group actions on von Neumann algebras, outer actions are important. But we note here that actions on Kac algebras are different from outer actions. In the following remark, we will give an example of an outer action which is not a group action on a Kac algebra.

REMARK 3.5. Consider a discrete ICC-group G and its character group $\chi(G)$. Let α be an action of $\chi(G)$ on a factor $L(G)$ defined by $\alpha_\chi(\lambda(g)) = \chi(g)\lambda(g)$. We know that α is an outer $\chi(G)$-action on $L(G)$.

But for a nontrivial character χ, if we take $g_0 \in G$ with $\chi(g_0) \neq 1$ then we have $\Gamma_s(\alpha_\chi(\lambda(g_0))) = \Gamma_s(\chi(g_0)\lambda(g_0)) = \chi(g_0)\lambda(g_0)\otimes\lambda(g_0)$ and $(\alpha_\chi \otimes \alpha_\chi)\Gamma_s(\lambda(g_0)) = (\alpha_\chi \otimes \alpha_\chi)(\lambda(g_0) \otimes \lambda(g_0)) = \chi(g_0)^2\lambda(g_0) \otimes \lambda(g_0)$, which implies that

$$\Gamma_s(\alpha_\chi(\lambda(g_0))) \neq (\alpha_\chi \otimes \alpha_\chi)\Gamma_s(\lambda(g_0)).$$

Thus α is not $\chi(G)$-action on a Kac algebra $\mathbb{K}_s(G)$.

Now we consider an investigation of the associated action $\hat{\alpha}$ on $\hat{M}$ of a G-action α on M.

It was shown in [DeC1] that the canonical implementation u_g of α_g also implements an automorphism $\hat{\alpha}_g$ of $\hat{M}$ for any $g \in G$. This $\hat{\alpha}$, so-called associated action of G on $\hat{M}$ (see Definition 2.9 in [DeC1]), allows us to define a normal injective homomorphism $\pi_{\hat{\alpha}} : \hat{M} \to \hat{M} \otimes L^\infty(G)$

and we get a G-action $\hat{\alpha}$ on $\hat{\mathbb{K}}$ with the crossed product Kac algebra $\hat{\mathbb{K}} \rtimes_{\hat{\alpha}} G$ (see [DeC1]). Here we give examples of group actions on Kac algebras.

EXAMPLE 3.6. For two locally compact groups K and H, let α be a K-action on $L(H)$ satisfying that for any $k \in K$ and $h \in H$, $\alpha_k(\lambda(h)) = \lambda(h')$ for some $h' \in H$. Then K-actions α and $\hat{\alpha}$ are K-actions on $\mathbb{K}_s(H)$ and $\mathbb{K}_a(H)$, respectively, with the same unitary implementation.

From Theorem 3.4, when α is an G-action on $\mathbb{K}$ we note that $\pi_\alpha(G(\mathbb{K}))$ is a subgroup of $G(\mathbb{K} \rtimes_\alpha G)$. Moreover, there exists a G-action on a group $G(\mathbb{K})$ which induces G-action β on a Kac algebra $\mathbb{K}_s(G(\mathbb{K}))$ with $\mathbb{K}_s(G(\mathbb{K})) \rtimes_\beta G \subset \mathbb{K} \rtimes_\alpha G$.

REMARK 3.7. Let G be a semi-direct product $H \rtimes_\beta K$ of locally compact groups H and K. It is well known that there exists K-actions α and $\hat{\alpha}$ on $L(H)$ and $L^\infty(H)$, respectively, induced by β (see [DeC1]).

(1) We know that $L(H) \rtimes_\alpha K = L(G)$ and $G(\mathbb{K}_s(H)) = H$ gives $\mathbb{K}_s(H) \rtimes_\alpha K = \mathbb{K}_s(G)$.
(2) We have $L^\infty(H) \otimes L^\infty(K) = L^\infty(G)$. But neither $\mathbb{K}_a(H) \otimes \mathbb{K}_a(K)$ nor $\mathbb{K}_a(H) \rtimes_{\hat{\alpha}} K$ is isomorphic to $\mathbb{K}_a(G)$.

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