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HOLOMORPHIC PRINCIPLE LINE BUNDLES OVER COMPLEX GROUPS

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1. Introduction

Let F be a complex line bundles over a complex manifold M. In [7], we investigated the properties of holomorphic line bundles F of cohomology groups for a complex torus. And also we know that the group of holomorphic line bundles on a q-dimensional complex torus with the first Chern class zero is a family of weakly pseudoconvex manifolds. K. H. Shon and H. R. Cho [6] obtained some properties of a family of weakly pseudoconvex manifolds. H. Kazamà and K. H. Shon [2,3] solved the $\bar{\partial}$-problem on a family of weakly pseudoconvex manifolds. T. Ueda [8] investigated some properties of a family of a compact complex curve with topologically trivial normal bundle. Recently H. Kazama, T. Ohta and K. H. Shon [1] obtained (non)vanishing and imbedding theorem on weakly complex spaces. In this paper, we obtain some properties of topological holomorphic line bundles with respect to a complex torus.

2. Topological principle line bundles

Definition 2.1. Let M be a topological space. M is said to have

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the structure of an n-dimensional complex manifold if there exists an atlas $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ of charts on M such that

(1) ϕ_i is a homeomorphism of U_i onto the open subset $\phi_i(U_i)$ of $\mathbf{C}^n$ for all $i \in I$.

(2) For all $i, j \in I$, $\phi_i \phi_j^{-1}$ is a biholomorphic map of $\phi_j(U_{ij})$ onto $\phi_i(U_{ij})$, where $U_{ij} = U_i \cap U_j$.

DEFINITION 2.2. Let M be a differentiable manifold. A differentiable manifold F is called a (complex) line bundle over M if it satisfies the following conditions:

(1) A C^∞ map $\pi : F \to M$ of F onto M is given.

(2) For every $p \in M, \pi^{-1}(p)$ is an n-dimensional $\mathbf{C}$ -vector space : $\pi^{-1}(p) \cong \mathbf{C}^n$, where n is independent of p

(3) For every $q \in M$, there exists a neighborhood U, $q \in U \subset M$, such that $\pi^{-1}(U) = U \times \mathbf{C}^n$, and that for any $p \in U$, $p \times \mathbf{C}^n$ is isomorphic to $\pi^{-1}(p)$ as an $\mathbf{C}$-vector space : $\pi^{-1}(p) \cong \{p\} \times \mathbf{C}^n$.

We denote the line bundle by $\pi : F \to M$ or just F. Let $\pi : M \times \mathbf{C}^n \to M$ denote projection on the first factor. Then $\pi : M \times \mathbf{C}^n \to M$ is a line bundle over M called the trivial line bundle over M. Now consider line bundles over a complex manifold. Let F be a line bundle over a complex manifold M. If the transitive functions $f_{jk}(p), j, k = 1, 2, \cdots$, are all holomorphic, F is called a holomorphic line bundle. Here by saying that $f_{jk}(p) = (f^\alpha_{jk\beta}(p))$ is holomorphic, we mean that each component $f^\alpha_{jk\beta}(p)$ is a holomorphic function of $p \in U_{jk}$. Then F is obtained by glueing up $U_k \times \mathbf{C}^n$ by identifying $(p, \zeta_j) \in U_j \times \mathbf{C}^n$ with

$$(p, \zeta_j) = (p, f_{jk}(p)\zeta_k) \in U_j \times \mathbf{C}^n \ ; \ F = \cup_j U_j \times \mathbf{C}^n.$$

From K. Kodaira [4] and A. Morrow and K. Kodaira [5], if $f_{jk}(p)$ is holomorphic, then the map $(p, \zeta_k) \to (p, \zeta_j)$ is biholomorphic. Let $e_1^*, e_2^*, \cdots, e_{q+1}^*$ be $q+1$ unit vectors of $\mathbf{C}^{q+1}$ and

$$v_i = (v_{i1}, v_{i2}, \cdots, v_{iq}) \in \mathbf{C}^q.$$

For any $v_{1\ q+1}, v_{2\ q+1}, \cdots, v_{q\ q+1} \in \mathbf{C}$, we let

$$\begin{aligned} v_1^* &:= (v_{11}, v_{12}, \cdots, v_{1q}, v_{1\ q+1}) \in \mathbf{C}^{q+1}, \\ v_2^* &:= (v_{21}, v_{22}, \cdots, v_{2q}, v_{2\ q+1}) \in \mathbf{C}^{q+1}, \\ &\cdots \\ v_q^* &:= (v_{q1}, v_{q2}, \cdots, v_{qq}, v_{q\ q+1}) \in \mathbf{C}^{q+1}. \end{aligned}$$

Then

$$\begin{aligned} \Gamma^* &:= Ze_1^* + \cdots + Ze_{q+1}^* + \cdots + Zv_1^* + \cdots + Zv_q^* \\ &= \{m_1e_1^* + \cdots + m_{q+1}e_{q+1}^* + n_1v_1^* + n_qv_q^* \ : \ m_i, n_i \in Z\} \end{aligned}$$

is a additive discrete subgroup of $\mathbf{C}^{q+1}$.

LEMMA 2.3. *The quotient group $\mathbf{C}^{q+1}/\Gamma^*$ is a non- compact abelian Lie group.*

Proof. Let $e_1^*, e_2^*, \cdots, e_{q+1}^*, v_1^*, v_2^*, \cdots, v_q^* \in \mathbf{C}^{q+1} \cong \mathbf{R}^{2q+2}$. Then there exists v_{q+1}^* in $\mathbf{C}^{q+1}$ such that

$$\{\mu v_{q+1}^*\} \subset \mathbf{C}^{q+1}, \mu = 1, 2, \cdots$$

and

$$\mu v_{q+1}^* + \Gamma^* \in \mathbf{C}/\Gamma^*.$$

Hence the set $\{\mu v_{q+1}^* + \Gamma^*\}$ is discrete. That is, there exist no convergent subsequence of the set. Thus, we complete the proof. □

Consider the projection

$$p : \mathbf{C}^{q+1} \to \mathbf{C}^q$$

satisfying $(z_1, z_2, \cdots, z_{q+1}) \mapsto (z_1, z_2, \cdots, z_q)$ and

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \to \mathbf{C}^q/p(\Gamma^*).$$

Then

$$p(\Gamma^*) = ze_1 + \cdots + ze_q + zv_1 + \cdots + zv_q = \Gamma.$$

Thus, we have

$$\mathbf{C}^q/p(\Gamma^*) = \mathbf{C}^q/\Gamma = \mathbf{T}^q$$

where $\mathbf{T}^q$ is a complex torus (see [7]). Therefore from Lemma 2.3 ,

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \longrightarrow \mathbf{T}^q$$

have a structure of principle line bundles. Let

$$\tilde{U}_i := \{z_i + \Gamma \in \mathbf{C}/\Gamma : \forall z_i \in U_i\}$$

is an open subset of $\mathbf{T}^1 = \mathbf{C}/\Gamma$. In the case of $\mathbf{C}^2$,

$$p^* : \mathbf{C}^2/\Gamma^* \longrightarrow \mathbf{T}^1,$$
$$\Gamma^* = \mathbf{Z}e_1^* + \mathbf{Z}e_2^* + \mathbf{Z}v_1^*,$$
$$p^{*-1}(\tilde{U}_i) = \{z + \Gamma^* : \forall z = (z_1, z_2) \in U_i \times \mathbf{C}\}.$$

Suppose that

$$\pi_i : \ p^{*-1}(\tilde{U}_i) \longrightarrow \tilde{U}_i \times \mathbf{C}^*$$

satisfying $\pi_i(z + \Gamma^*) = (z_1 + \Gamma, exp\ 2\pi\sqrt{-1}z_2)$ where $z = (z_1, z_2)$ and $z_1 \in U_i$.

LEMMA 2.4. *π_i is a well defined biholomorphic onto mapping.*

Proof. Suppose that

$$z + \Gamma^* = \tilde{z} + \Gamma^* \in p^{*-1}(\tilde{U}_i)$$

where $z = (z_1, z_2), \tilde{z} = (\tilde{z}_1, \tilde{z}_2), z_1, \tilde{z}_1 \in U_i$. Since $z - \tilde{z} \in \Gamma^*$, there exists $m_i \in \mathbf{Z}$ such that

$$z - \tilde{z} = m_1 e_1^* + m_2 e_2^* + m_3 v_1^*.$$

Thus, we have

$$z_1 - \tilde{z}_1 = m_1 + m_3 v_{11},$$
$$z_2 - \tilde{z}_2 = m_2 + m_3 v_{12}.$$

Since $z_1, \tilde{z}_1 \in U_i$, we have

$$m_1 = m_3 = 0.$$

Hence $z_2 - \tilde{z}_2 = m_2$. And

$$\begin{aligned}&(z_1 + \Gamma, exp\ 2\pi\sqrt{-1}z_2)\\ =&(\tilde{z}_1 + \Gamma, exp\ 2\pi\sqrt{-1}(\tilde{z}_2 + m_2))\\ =&(\tilde{z}_1 + \Gamma, exp\ 2\pi\sqrt{-1}\tilde{z}_2).\end{aligned}$$

□

THEOREM 2.5.

$$p^* : \mathbf{C}^{q+1}/\Gamma^* \longrightarrow \mathbf{T}^q$$

is a topological trivial holomorphic principle line bundle.

Proof. From Lemma 2.4, it is a holomorphic principle line bundle. Now we prove that it is topological trivial. We consider exact sequences:

$$0 \to \mathbf{Z} \to C \xrightarrow{\Phi} C^* \to 0,$$
$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where $\Phi(\cdot) = exp2\pi\sqrt{-1}(\cdot)$, C is the sheaf of germs of continuous functions, C^* is the nonzero sheaf, $\mathcal{O}$ is the sheaf of germs of complex - valued C^∞ functions and $\mathcal{O}^*$ is the nonzero sheaf. Hence we have the long exact sequences

$$\cdots \to H^1(\mathbf{T}^q, C) \to H^1(\mathbf{T}^q, C^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots,$$
$$\cdots \to H^1(\mathbf{T}^q, \mathcal{O}) \to H^1(\mathbf{T}^q, \mathcal{O}^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots.$$

Since $H^1(\mathbf{T}^q, \mathcal{O}^*)$ is the group of all holomorphic line bundles on $\mathbf{T}^q$, the Chern class $H^2(\mathbf{T}^q, \mathbf{Z})$ is zero. Therefore we complete the proof.□

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