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INDEX AND STABLE RANK OF C^*-ALGEBRAS

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ABSTRACT. We show that if the stable rank of B^α is one, then the stable rank of B is less than or equal to the order of G for any action of a finite group G. Also we give a short proof to the known fact that if the action of a finite group on a C^*-algebra B is saturated then the canonical conditional expectation from B to B^α is of index-finite type and the crossed product C^*-algebra is isomorphic to the algebra of compact operators on the Hilbert B^α-module B.

1. Introduction

We recall notations and properties on the index for C^*-subalgebras from [8]. Let B be a unital C^*-algebra and A a C^*-subalgebra of B with the same unit I. Let E be a faithful conditional expectation of B onto A. Then E is called of *index-finite type* if there exists a finite set $\{u_1, u_2, \dots, u_n\} \subset B$ for E, such that

$$x = \sum_{i=1}^{n} u_i E(u_i^* x) = \sum_{i=1}^{n} E(x u_i) u_i^*.$$

When E is of index-finite type, the index of E is defined by

$$\text{Index } E = \sum_{i=1}^{n} u_i u_i^*.$$

The value Index E does not depend on the choice of $\{u_i | i = 1, \dots, n\}$ and Index E is in the center of B. When $A \subset B$ is a factor-subfactor pair, Index E coincides with Kosaki's index [2].

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In this note, by using the index we show that if the stable rank of B^α is one, then the stable rank of B is less than or equal to the order of G for any action α of a finite group G. Also we give a short proof that if the action of a finite group G on a C^*-algebra B is saturated then the canonical conditional expectation E from B to B^α is of index-finite type and the crossed product C^*-algebra $B \times_\alpha G$ is isomorphic to the algebra of compact operators on the Hilbert B^α-module B which was proved in [3] for the Hopf algebras.

2. Preliminaries

Let $E : B \to A$ be a faithful conditional expectation. Put $\mathcal{E}_0 = B = B_A$, where B_A means that B is a right A-module. Then $\mathcal{E}_0$ is a pre-Hilbert module over A with A-valued inner product $< \eta(x), \eta(y) >= E(x^*y)$ for $x, y \in B$, where we use the notation $\eta(x) \in \mathcal{E}_0$ for $x \in B$. Let $\mathcal{E}$ be the completion of $\mathcal{E}_0$ by the norm

$$\|\eta(x)\| = \| < \eta(x), \eta(x) > \|^{1/2} = \|E(x^*x)\|^{1/2}.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over A. Let $\mathcal{L}_A(\mathcal{E})$ be the set of all A-module homomorphism $T : \mathcal{E} \to \mathcal{E}$ with an adjoint A-module homomorphism $S : \mathcal{E} \to \mathcal{E}$ such that

$$< T\xi, \zeta >=< \xi, S\zeta > .$$

We denote S as T^*. Then $\mathcal{L}_A(\mathcal{E})$ is a C^*-algebra with the usual norm $\|T\| = sup\{\|T\xi\| : \|\xi\| = 1\}$. For $\xi, \zeta \in \mathcal{E}$, let $\theta_{\xi,\zeta}$ be the "rank one" operator defined by $\theta_{\xi,\zeta}(\gamma) = \xi < \zeta, \gamma >$. Let $\mathcal{K}_A(\mathcal{E})$ be the norm closure of the linear span of $\{\theta_{\xi,\zeta} : \xi, \zeta \in \mathcal{E}\}$, the algebra of "compact operators". For $b \in B$, define $\lambda(b) \in \mathcal{L}_A(\mathcal{E})$ by

$$\lambda(b)\eta(x) = \eta(bx) \text{ for } x \in B.$$

Then $\lambda : B \to \mathcal{L}_A(\mathcal{E})$ is an injective *-homomorphism. For $x \in B$, put $e_A\eta(x) = \eta(E(x))$. Then e_A can be extended to a bounded linear operator on $\mathcal{E}$ and it is a projection in $\mathcal{L}_A(\mathcal{E})$. Let $C_r^* < B, e_A >$ be the closure of the linear span of $\{\lambda(x)e_A\lambda(y) \in \mathcal{L}_A(\mathcal{E}) | x, y \in B\}$. Note that $\lambda(x)e_A\lambda(y^*)$ is the rank one operator $\theta_{\eta(x),\eta(y)}$.

Consider a C^*-algebra B and a finite group G. Let $\alpha : G \to Aut(B)$ be an action. Put $A = B^\alpha$ and let $E : B \to A$ be the conditional expectation given by

$$E(x) = \frac{1}{\#G} \sum \alpha_g(x) \text{ for } x \in B.$$

Since $\alpha_g E = E$ for $g \in G$, we can define unitaries $u_g \in \mathcal{L}_A(\mathcal{E})$ for $g \in G$ by

$$u_g \eta(x) = \eta(\alpha_g(x)) \text{ for } x \in B.$$

Then we have the covariant relation

$$u_g b u_g^* = \alpha_g(b) \text{ for } b \in B \subset \mathcal{L}_A(\mathcal{E}).$$

By the universality of the crossed product, there is a *-homomorphism $\psi : B \times_\alpha G \to \mathcal{L}_A(\mathcal{E})$ such that

$$\psi(\sum_{g \in G} b_g \lambda_g) = \sum_{g \in G} b_g u_g.$$

Put $p = \frac{1}{\#G}(\sum_{g \in G} \lambda_g)$. Then $\psi(p) = e_A$. In fact,

$$\begin{aligned} e_A \eta(b) = \eta(E(b)) = \eta(\frac{1}{\#G}(\sum_{g \in G} \alpha_g(b) = \frac{1}{\#G}(\sum_{g \in G} \eta(\alpha_g(b)\\ = \frac{1}{\#G}(\sum_{g \in G} u_g(\eta(b)) = \psi(p)\eta(b). \end{aligned}$$

Since $\psi(xpy) = x e_A y$ for all $x, y \in B$, we have $\psi(B \times_\alpha G) \supset \mathcal{K}_A(\mathcal{E}) = C_r^* < B, e_A >$.

3. Saturated action of a finite group

DEFINITION 3.1 ([4]). *Let B be a C^*-algebra and $\alpha : G \to Aut(B)$ be an action. For $x \in B$ put $\tilde{x} = \sum_g \alpha_g(x)\lambda_g \in B \times_\alpha G$. Then α is called saturated* if the elements $\tilde{x}^* \tilde{y}$, for $x, y \in B$, span a dense subspace of $B \times_\alpha G$.

Then

$$\tilde{x} = \sum_g \alpha_g(x)\lambda_g = \sum_g \lambda_g x \lambda_g^* \lambda_g = (\sum_g \lambda_g)x = (^\# G)px$$

and

$$\tilde{x}^* \tilde{y} = (^\# G)^2 x^* py.$$

Let L be the linear span of $\{\tilde{x}^*\tilde{y} \in B \times_\alpha G; x, y \in B\}$. Then L is an algebraic ideal of $B \times_\alpha G$. Thus

$$\alpha \text{ is saturated } \iff 1 \in \bar{L} \iff 1 \in L \iff L = B \times_\alpha G.$$

For the following theorem there is a general result for finite dimensional Hopf algebras which include finite groups in [3] but the construction is more complicated because of the greater generality. We give here a short proof for finite groups without the term 'Hopf algebra'.

THEOREM 3.2. *Let B be a C^*-algebra and $\alpha : G \to Aut(B)$ be an action of a finite group G. Define a conditional expectation $E : B \to A = B^\alpha$ by $E(b) = \frac{1}{\#G}\sum_g \alpha_g(b)$. If α is saturated, then E is of index-finite type and $B \times_\alpha G \cong C_r^* < B, e_A >= \mathcal{L}_A(\mathcal{E})$.*

Proof. It is shown in [8] that E is of index-finite type. Now Consider the *-homomorphism $\psi : B \times_\alpha G \to \mathcal{L}_A(\mathcal{E})$ such that

$$\psi(\sum_{g \in G} b_g \lambda_g) = \sum_g b_g u_g.$$

Since

$$\psi(\tilde{x}^* \tilde{y}) = \psi((\#G)^2 x^* py) = (\#G)^2 x^* e_A y,$$

$\psi(\bar{L}) = \mathcal{K}_A(\mathcal{E}) = C_r^* < B, e_A >$. Since α is saturated, $1 \in \bar{L}$. Therefore $C_r^* < B, e_A >$ has an identity. This shows that α is surjective as shown in [8]. Next we show that α is injective. Suppose that $\sum_i x_i e_A y_i = 0$. Then for any $b \in B$,

$$\begin{aligned}\sum_i (x_i e_A y_i)(b) &= \sum_i x_i e_A \eta(y_i b) = \sum_i x_i \eta(E(y_i b)) = \sum_i \eta(x_i E(y_i b)) \\ &= \eta(\sum_i (x_i E(y_i b))) = 0.\end{aligned}$$

So

$$\sum_g \sum_i x_i \alpha_g(y_i b) = (^{\#}G) \sum_i (x_i E(y_i b)) = 0.$$

Then for any $b, c \in B$,

$$\begin{aligned}
(\sum_i x_i p y_i) bpc &= \frac{1}{\#G}[\sum_i x_i\{(\sum_g \lambda_g) y_i b\}]pc \\
&= \frac{1}{\#G}[\sum_i x_i\{(\sum_g \alpha_g(y_i b)\lambda_g\}]pc \\
&= \frac{1}{\#G}[\sum_i \sum_g x_i \alpha_g(y_i b)\lambda_g]pc \\
&= \frac{1}{\#G}[\sum_g (\sum_i x_i \alpha_g(y_i b))\lambda_g]pc \\
&= \frac{1}{\#G}\sum_g (\sum_i x_i \alpha_g(y_i b))\lambda_g pc \\
&= \frac{1}{\#G}\sum_g (\sum_i x_i \alpha_g(y_i b)) pc \\
&= 0.
\end{aligned}$$

Since α is saturated this implies that $(\sum_i x_i p y_i) = 0$. This shows that ψ is an isomorphism. □

Let (B, G, α) be a C^*-dynamical system. Recall that the action is *topologically free* if for any $t_1, \ldots. t_n \in G \backslash \{e\}, \cap_{i=1}^n \{x \in \hat{B} | t_i x \neq x\}$ is dense in $\hat{B}$ (the spectrum of B), where $tx(b) = x(\alpha_t(b))$.

COROLLARY 3.3. *Let (B, G, α) be a topologically free dynamical system with G finite and B has no nontrivial α-invariant ideals. Then the canonical conditional expectation $E : B \to B^\alpha = A$ is of index-finite type and $B \times_\alpha G \cong \mathcal{K}_A(\mathcal{E}) = \mathcal{L}_A(\mathcal{E})$.*

Proof. By [1], $B \times_\alpha G$ is simple. Since $B \times_\alpha G$ is simple, α is saturated. Hence we get the conclusion. □

4. Stable ranks and finite groups

Let B be a unital C^*-algebra and $Lg_n(B)$ be the set of n-tuples $(b_1, \dots, b_n)$ that generate B as a left ideal. The *stable rank*, $sr(B)$, is defined to be the least integer n for which $Lg_n(B)$ is dense in B^n (see [5]). If $Lg_n(B)$ is never dense, set $sr(B) = \infty$. It is easy to show that $sr(B) = 1$ if and only if the set of invertible elements of B is dense in B. A C^*-algebra B is said to be *purely infinite* if every nonzero hereditary C^*-subalgebra of B has an infinite projection. It is not known whether every finite simple C^*-algebra is stably finite. It is well-known that if the stable rank of a simple C^*-algebra is finite then it is stably finite. Given a C^*-dynamical system (B, G, α) there are many interesting results concerning the stable rank of B^α and that of $B \times_\alpha G$.

Now we consider a relation between the stable rank of a C^*-algebra B and that of the fixed point subalgebra B^α.

LEMMA 4.1 ([8]). *The following are equivalent:*

(1) *$E : B \to A$ is of index-finite type*

(2) *$C_r^* < B, e_A >$ has an identity and there is a constant $c > 0$ such that*

$$E(x^*x) \geq cx^*x \text{ for } x \in B$$

LEMMA 4.2. *Let (B, G, α) be a C^*-dynamical system with G, a finite group of order n. If $B \times_\alpha G$ has stable rank one, then stable rank of B, $sr(B) \leq n$.*

Proof. Let $(b_{g_1}, \dots, b_{g_n}) \in B^n$ and $y = \sum_{i=1}^n b_{g_i}\lambda_{g_i} \in B \times_\alpha G$. Consider the canonical conditional expectation E from $B \times_\alpha G$ to B given by

$$E(\sum_g a_g \lambda_g) = a_e.$$

Note that $\{\lambda_g | g \in G\}$ is a quasi-basis for E and hence E is of index-finite type. Since $sr(B \times_\alpha G) = 1$, we can approximate y by an invertible $x = \sum_{i=1}^n c_{g_i}\lambda_{g_i}, c_{g_i} \in B$. Clearly, $(c_{g_1}, \dots, c_{g_n})$ is close to $(b_{g_1}, \dots, b_{g_n})$. By Lemma 4.1, there is a constant $c > 0$ such that

$$cx^*x \leq E(x^*x) = \sum_g c_g^* c_g$$

Hence $\sum_g c_g^* c_g$ is invertible in B. This shows that $(c_{g_1}, \ldots, c_{g_n}) \in Lg_n(B)$ completing the proof. □

THEOREM 4.3. *Let B be a C^*-algebra and $\alpha : G \to Aut(B)$ be an action of a finite group G. If $sr(B^\alpha) = 1$, then $sr(B) \leq n$, where n is the order of G.*

Proof. Since it was shown in [7] that B^α is stably isomorphic to $B \times_\alpha G$ it follows from the fact that $sr(B^\alpha) = 1$ if and only if $sr(B \times_\alpha G) = 1$. □

REMARK 4.4. It is shown in [6] that if (B, G, α) is a C^*-dynamical system where G is compact abelian, then

$$min\{sr(B), 2\} \leq sr(B \times_\alpha G) \leq sr(B^\alpha).$$

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