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ON THE REGULARITY AND THE HOLOMORPHICAL REGULARITY

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Abstract. In this paper, we introduce the regularity, the hyperexactness and the hyperregularity, and we study on the extensions of regularity and the holomorphical regularity of the bounded linear operators.

1. Introduction

Throughout this paper, we suppose that X is a complex Banach space and write $\mathrm{BL}(X)$ for the set of all bounded linear operators on X. We denote, for $T \in \mathrm{BL}(X)$,

$$\mathrm{comm}(T) = \{S \in \mathrm{BL}(X) |\ ST = TS\},$$

$$\mathrm{comm}^{-1}(T) = \{S \in \mathrm{BL}(X) |\ ST = TS, S \text{ is invertible}\},$$

$T^\infty(X) = \cap_{n=1}^{\infty} T^n(X)$ for the hyperrange, $T^{-\infty}(0) = \cup_{n=1}^{\infty} T^{-n}(0)$ for the hyperkernel of T. An operator $T \in \mathrm{BL}(X)$ is called regular if there is $T' \in \mathrm{BL}(X)$ such that $T = TT'T$.

We say that an operator $T \in \mathrm{BL}(X)$ is hyperexact if $T^{-1}(0) \subseteq T^\infty(X)$, and hyperregular if T is regular and hyperexact, and holomorphically regular if there is $\delta > 0$ and a holomorphic mapping $T'_\lambda : \{\lambda \in \mathbb{C} |\ |\lambda| < \delta\} \to \mathrm{BL}(X)$ for which $T - \lambda I = (T - \lambda I)T'_\lambda(T - \lambda I)$ for each $|\lambda| < \delta$. We call $T \in \mathrm{BL}(X)$ proper if $\mathrm{core}(T) : X/T^{-1}(0) \to cl(T(X))$ is invertible with $\mathrm{core}(T)(x+T^{-1}(0)) = Tx$ for $x+T^{-1}(0) \in X/T^{-1}(0)$. In this paper, we find the necessary conditions for the finite sum of bounded linear operators to be holomorphically regular.

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2. Preliminaries

Let $T + S \in \mathrm{BL}(X)$ be onto. We first observe that $T^n(X) \subseteq (T + S)^n(X)$ for each $n \in \mathbb{N}$.

LEMMA 1. *Let X be a complex Banach space and let $T = TT'T$ be hyperregular. If $S \in comm(T)$ with $\|T'S\| < 1$, then $T - S$ is regular.*

Proof. This follows from the proof of [1, theorem 9]. □

THEOREM 1. *Let X be a Hilbert space and let $S \in comm^{-1}(T)$ for $S, T \in BL(X)$. If $T+S \in BL(X)$ is onto, then $T+S$ is holomorphically regular.*

Proof. Since X is a Hilbert space and $T + S$ is onto, we have that $T + S$ is proper on X, and that $(T + S)^{-1}(0)$ and $(T + S)(X) = cl(T + S)(X)$ are complemented, respectively. This means that $T + S$ is regular([2,(3.8.2)]). For each $x \in (T + S)^{-1}(0)$, $(T + S)(x) = 0 \Longleftrightarrow Tx = -Sx$. Since S is invertible, we have

$$\begin{aligned} x &= -S^{-1}Tx = -S^{-1}T(-S^{-1}Tx) \\ &= T^2(-S^{-2})x = \cdots = T^n(-S^{-n})x \subseteq T^n(X) \end{aligned}$$

for each $n \in \mathbb{N}$. Thus $(T + S)^{-1}(0) \subseteq T^\infty(X)$. Let $T + S$ be onto and let $T^n(X)$ be a subspace of X. Then

$$(T + S)(T^n(X)) = T^n(X) = \cdots = (T + S)^n(T^n(X)) \subseteq (T + S)^n(X)$$

for each $n \in \mathbb{N}$. So, we have $T^\infty(X) \subseteq (T + S)^\infty(X)$. □

THEOREM 2. *Let X be a complex Banach space and let $S \in comm(T)$ for $S, T \in BL(X)$. If T is hyperregular with the generalized inverse $T' \in BL(X)$, $\|T'S\| < 1$, $T - S$ is onto, and $X/T^n(X)$ is finite dimensional for each $n \in \mathbb{N}$, then $T - S$ is holomorphically regular.*

Proof. Since T is hyperregular and $S \in \mathrm{comm}(T)$ with $\|T'S\| < 1$, we have that $T - S$ is regular(Lemma 1). From the assumption $T - S$ is onto and $X/T^n(X)$ is finite dimensional we have that

$$(T - S)^{-1}(0) \subseteq T^n(X) = (T - S)(T^n(X)) = \cdots = (T - S)^n(T^n(X))$$

for each $n \in \mathbb{N}$. This means that $(T - S)^{-1}(0) \subseteq (T - S)^\infty(X)$. □

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