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SOME CHARACTERIZATIONS OF KRULL DOMAINS

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ABSTRACT. We will find sufficient conditions for a Mori domain to be a Krull domain.

1. Introduction

Many of characterizations of Dedekind domains have t-operation analogues for Krull domains (see [8], [10]). Thus from the well-known characterizations of Dedekind domains, we can deduce new characterizations of Krull domains. In this paper, in spirit of [6, Theorem 37.8 and Theorem 38.1], we find sufficient conditions for a Mori domain to be a Krull domain. In particular, we give examples which show that the conditions of Theorem 6 are the best possible.

Throughout this paper, R will denote a commutative integral domain with identity and K its quotient field. Let $F(R)$ be the set of nonzero fractional ideals of R. For each $A \in F(R)$, $A_v = (A^{-1})^{-1}$ and $A_t = \cup\{J_v : J$ is a finitely generated subideal of $A\}$. If $A_v = A$ (resp. $A_t = A$) then A is said to be a divisorial ideal (resp. t-ideal). We have $A \subset A_t \subset A_v$, so that every divisorial ideal is a t-ideal. If $A_t = J_t$ for some finitely generated subideal of A, A_t is said to be of finite type. R is called a Mori domain if each t-ideal of R is of finite type, or equivalently, accending chain condition on t-ideals holds. By a chain of prime t-ideals of R we mean a finite strictly increasing sequence $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$; the length of the chain is n. We define the t-dimension of R, denoted by t-dim$R$, to be the supremum of the lengths of all chains of prime t-ideals in R.

Unexplained terminology is standard, as in [6] or [9].

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2. Main results

For a technical reason, we assume that R is not a field, i.e., $R \subsetneq K$. A maximal t-ideal of R is a proper t-ideal of R which is maximal among proper t-ideals of R. It is easy to see by Zorn's lemma that each maximal t-ideal is prime and the set of maximal t-ideals is not empty.

In [1], D. D. Anderson shows that for a proper ideal I of the ring R with identity if each prime ideal minimal over I is finitely generated then the number of prime ideals which are minimal over I is finite. By the same way as Anderson's proof [1] and the fact that if a prime ideal P of R is minimal over a t-ideal then P is also a t-ideal, we have the following useful result.

LEMMA 1. *Let I be a t-ideal of R. If each prime ideal P of R which is minimal over I is of finite type, i.e.,there is a finitely generated subideal A of P such that $A_t = P$, then the number of prime ideals minimal over I is finite.*

LEMMA 2. *If $\{M_\lambda\}$ is the set of maximal t-ideals of R, then $R = \cap R_{M_\lambda}$.*

Proof. see [6, Ex. 22, page 52]. □

DEFINITION 3. Let R be an integral domain and $X^1(R)$ the set of nonzero minimal prime ideals of R. A domain R is called a *Krull domain* if

1. $R = \cap_{P \in X^1(R)} R_P$,
2. R_P is a rank one DVR for each $P \in X^1(R)$, and
3. for each $0 \neq a \in R$, the set of prime ideals of $X^1(R)$ containing a is finite.

THEOREM 4. *(cf. [6, Theorem 38.1]) Let R be a Mori domain which is not a field and $\{M_\alpha\}$ the set of maximal t-ideals of R, then the following conditions are equivalent.*

1. *R is a Krull domain.*
2. *Each M_α is t-invertible, i.e., $(M_\alpha M_\alpha^{-1})_t = R$.*
3. *$\{(M_\alpha^n)_t\}$ is the set of M_α- primary ideals and for each α, there is a prime ideal $P_\alpha \subsetneq M_\alpha$ such that there are no prime ideals properly between P_α and M_α.*

Proof. (1) $\Longrightarrow$ (2) [8, Theorem 3.6].

(2) $\Longrightarrow$ (3) [8, Theorem 2.2].

(3) $\Longrightarrow$ (1) For a maximal t-ideal M of R, $(M^k R_M)_t = ((M^k)_t R_M)_t = (M^k)_t R_M$ [7, Proposition 1.1]. Since MR_M is a maximal ideal of R_M, $M^k R_M = (MR_M)^k$ is an MR_M-primary ideal. So $M^k R_M \cap R$ is M-primary. So $M^k R_M \cap R = (M^l)_t$ for some positive integer l. So $M^k R_M = (M^k R_M \cap R) R_M = (M^l)_t R_M = (M^l R_M)_t$. Thus $M^k R_M = (M^l R_M)_t = ((M^l R_M)_t)_t = (M^k R_M)_t$. Thus $\{M^k R_M\}_{k=1}^{\infty}$ is the set of MR_M-primary ideals. Let P be a prime ideal of R such that $P \subsetneq M$ and there are no prime ideals properly between P and M. Since each $MR_M/(PR_M)$-primary ideal is of the form $A/(PR_M)$ where A is a MR_M-primary ideal containing PR_M and the number of $MR_M/(PR_M)$-primary ideals is infinite, $PR_M \subseteq M^k R_M$ and $M^k R_M \neq M^{k+1} R_M$ for each positive integer k. Since R_M is a Mori domain, $PR_M = 0$ [8, Theorem 2.1] and hence $P = 0$. Thus R_M is a rank one DVR. By Lemmas 1 and 2, R is a Krull domain. □

A domain R is said to be a Prüfer v-multiplication domain (PVMD) if each finitely generated ideal I of R is t-invertible, i.e., $(II^{-1})_t = R$, or equivalently, R_P is a valuation domain, for each maximal t-ideal P. Recall from [9, page26] that a domain R is called an S-domain if for every height one prime ideal P, the expansion $P[X]$ of P to the polynomial ring $R[X]$ also has height one.

LEMMA 5. *Let R be a domain of t-$dimR \leq 1$, then R is an integrally closed S-domain if and only if R is a PVMD.*

Proof. ($\Longrightarrow$) If P is a maximal t-ideal, R_P is an one dimensional integrally closed domain. Since R is an S-domain, $ht(P[X]) = 1$. So $dimR_P[X] = dim(R_P[X]_{PR_P[X]}) + 1 = dim(R[X]_{P[X]}) + 1 = ht(P[X]) + 1 = 2$. Since R_P is integrally closed, R_P is a valuation domain [6, Proposition 30.14].

($\Longleftarrow$) If P is a prime ideal of $ht(P) = 1$, R_P is a valuation domain. So $dim(R_P[X]) = 2$ and hence $ht(P[X]) = ht(PR_P[X]) = 1$. □

THEOREM 6. *(cf. [6, Theorem 37.8]) A domain R is a Krull domain if (and only if) R is an integrally closed Mori domain of t-$dimR = 1$ and R is an S-domain.*

Proof. By Lemma 5, R is a PVMD. Since R is a Mori domain, R_P is a rank one DVR for each maximal t-ideal P. By Lemmas 1 and 2, R is a Krull domain. □

In Theorem 6, the hypothesis that R is an S-domain is necessary. To see this, we give an example of an integrally closed Mori domain R of t-dim$R = 1$, which is not a Krull domain.

EXAMPLE 7. Let C (resp. Q) be the field of complex (resp. rational) numbers and $\overline{Q}$ the algebraic closure of Q in C. Then the subring $D = \overline{Q} + XC[|X|]$ of the power series ring $C[|X|]$ is an integrally closed Mori domain of $dimD = 1$ [4, Theorem 3.2]. But D is not a valuation domain [5, Theorem 2.1(h)]. Thus D is not a Krull domain.

EXAMPLE 8. Let $\Re$ be the field of real numbers and $R = \Re[|x, y|] = \Re + M$, where $M = (x, y)$, the power series ring over $\Re$. Let $\overline{Q}$ be the algebraic closure of the field Q of rational numbers in $\Re$. Let $D = \overline{Q} + M$, then

1. D is integrally closed,
2. D is a Mori domain of t-dim$D = 2$ and D satisfies Krull's principal ideal theorem and
3. D is an S-domain.

Proof. 1. Since $\overline{Q}$ is the integral closure of Q in $\Re$, D is integrally closed [5, Theorem2.1.(b)].

2. By [9, Theorem 71 and Theorem 72], R is a Noetherian UFD of $dimR = 2$ and M is the unique maximal ideal of R. By [2, Proposition 3.8], $Spec(R) = Spec(D)$. So D is a Mori domain [4, Theorem 3.2] and D satisfies Krull's principal ideal theorem [3, Corollary 3.2]. By [2, Proposition 3.23], M is a t-ideal of D. Since $ht(M) = 2$, t-dim$R = 2$.
3. If P is a prime ideal of D of $ht(P) = 1$, $R \subseteq D_P$. For if $m \in M - P$, $r = (rm)\frac{1}{m} \in D_P$ for each $r \in R$. Since $Spec(R) = Spec(D)$, $R_P \subseteq (D_P)_{(R-P)} = D_P$. Since R_P is a rank one DVR, $R_P = D_P$. Thus the polynomial ring $D_P[X] = R_P[X]$ is of dimension 2, and $ht(P[X]) = ht(PD_P[X]) = 1$. So D is an S-domain. □

Example 7 and Example 8 show that the hypothesis in Theorem 6 for an integrally closed Mori domain to be a Krull domain cannot be weakened.

REMARK 9. In Theorem 6, the condition that R is a Mori domain can be replaced by the assumption that each prime t-ideal is of finite type. For if P is a prime t-ideal, there is a finitely generated subideal I of P such that $P = I_t$. So $PR_P = I_t R_P = (I_t R_P)_t = (IR_P)_t$. Since R_P is a valuation domain, PR_P is principal and so R_P is a local PID.

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