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THE COMPUTATION OF MULTIVARIATE B-SPLINES WITH APPLICATION TO SURFACE APPROXIMATIONS

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Abstract

In spite of the well developed theory and the practical use of the univariate B-spline, the theory of multivariate B-spline is very new and waits its practical use. We compare in this article the multivariate B-spline approximation with the polynomial approximation for the surface fitting. The graphical and numerical comparisons show that the multivariate B-spline approximation gives much better fitting than the polynomial one, especially for the surfaces which vary very rapidly.

1. Introduction

The theory of univariate spline is well developed and widely used in one dimensional data fitting or curve fitting. The possibility of the theory of multivariate B-spline was first noted by C. de Boor in [6]. The theory has been developed since then by C.A.Micchelli, W.Dahmen and others in a series of papers [3],[12],[13],$\cdots$ etc, but its application cannot be seen in the literature and in practice.

In this paper we consider the data fitting or surface fitting by the bivariate B-spline in the least squares sense and compare with the polynomial approximations. Our graphical and numerical results show that the bivariate B-spline approximation gives much better fitting than the polynomial one (see §6). We used the computer HP-9000 for the graphical representations of approximated surfaces and the calculations of errors.

In §2, we review the univariate spline and the basic properties of multivariate B-spline. In §3,§4, we describe a spline space which is needed for least squares approximation and the least squares method. The information needed to approximate data are given. In §5,§6, we describe our approximation schemes and compare each other for the test functions by means of visual representations and numerical errors. The analysis and conclusion are given. We give some graphs of basis functions of multivariate B-spline.

For convenience, let us fix some notations to be used throughout paper. The elements of the Euclidean space $R^s, s \geq 1$, are denoted by x,z,$\cdots$. The superscripts will be used to enumerate vectors $x^j, j = 1, 2, \cdots$. We will denote by x_i or x_i^j, the i-th component of x, x^j, respectively. We set $x^\beta = x^{\beta_1} \cdots x^{\beta_s}$. $|\beta| = \beta_1 + \cdots + \beta_s$. In addition

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$vol_s(A)$, $\mathcal{X}_A(x)$, $[A]$, $|A|$ denote the s-dimensional volume, the indicator function, the convex hull and the cardinality of a given set A, respectively.

2. The multivariate B-spline

Let $t_0, \cdots, t_n$ be any real numbers. Then $[t_0, \cdots, t_n]$g will denote the divided difference of g at $t_0, \cdots, t_n$. For the distinct points $t_0, \cdots, t_n$ the divided difference $[t_0, \cdots, t_n]g$ is given by the formula

$$[t_0, \cdots, t_n]g = \sum_{j=0}^{n} \frac{g(t_j)}{\prod_{i \neq j}(t_j - t_i)} \tag{2.1}$$

One way to define the univariate B-spline $M(t|t_0, \cdots, t_n)$ is by means of the formula,[1],

$$M(t|t_0, \cdots, t_n) = n![t_0, \cdots, t_n](\cdot - t)_+^{n-1}, \tag{2.2}$$

where the truncated function $(\cdot - t)_+$ is defined as

$$(x-t)_+ = \begin{cases} x - t, & x > t \\ 0, & \text{otherwise.} \end{cases}$$

As the n-th order divided difference of an n-th order truncated power, the univariate B-spline $M(t|t_0, \cdots, t_n)$ can be seen to be a nonnegative piecewise polynomial of degree n-1 supported on $[t_0, t_n]$. If the knots $t_0, \cdots, t_n$ are distinct, then by (2.1) $M(t|t_0, \cdots, t_n)$ has an explicit form

$$M(t|t_0, \cdots, t_n) = n! \sum_{j=0}^{n} \frac{(t_j - t)_+^{n-1}}{\prod_{i \neq j}(t_j - t_i)} \tag{2.3}$$

Although (2.3) may be used to evaluate the B-spline, the following recurrence relation is preferred for its numerical stability and efficiency in computation:

$$\begin{aligned} &M(t|t_0, \cdots, t_n) = \\ &\frac{n}{n-1}\left(\frac{t_n - t}{t_n - t_0} M(t|t_1, \cdots, t_n) + \frac{t - t_0}{t_n - t_0} M(t|t_0, \cdots, t_{n-1})\right) \end{aligned} \tag{2.4}$$

This important formula is due to de Boor, Cox and Mansfield [6]. According to the following formula,[6],

$$\begin{aligned} &\frac{d}{dt} M(t|t_0, \cdots, t_n) = \\ &\frac{n}{t_n - t_0}\left(M(t|t_0, \cdots, t_{n-1}) - M(t|t_1, \cdots, t_n)\right), \end{aligned} \tag{2.5}$$

the derivative of a B-spline can be computed in terms of lower order B-splines.

2.1. **The definition of the multivariate B-spline.**

Let $S^n = \{(\tau_0, \cdots, \tau_n) : \tau_j \geq 0, \sum_{j=0}^{n} \tau_j = 1\}$ be the standard n-simplex. Let $t_0, \cdots, t_n$ be a knot sequence in the real line. Then the divided difference $[t_0, \cdots, t_n]$g has the following Hermite-Gennochi formula:

$$[t_0, \cdots, t_n]g = \int_{S^n} g^{(n)}(t_0\tau_0 + \cdots + t_n\tau_n) d\tau_1 \cdots d\tau_n \tag{2.6}$$

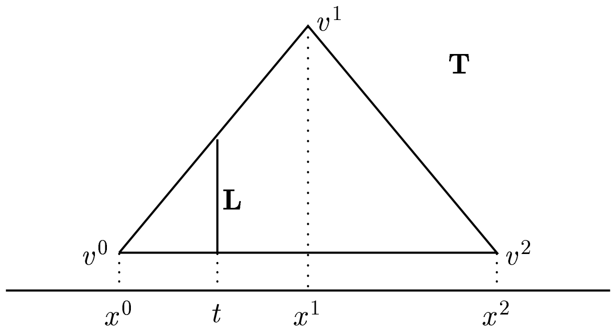


FIGURE 1

provided that g is n times continuously differentiable, [13]. By (2.2) and (2.6) we have

$$\int_{-\infty}^{\infty} g(t)M(t|t_0,\cdots,t_n)dt = n!\int_{S^n} g(t_0\tau_0+\cdots+t_n\tau_n)d\tau_1\cdots d\tau_n, \tag{2.7}$$

for any $g \in L^1_{loc}(R)$, the space of locally integrable functions on R. The formula (2.7) is used to define the s-dimensional B-spline.

Let $K = \{x^0,\cdots,x^n\}$ be any set of knots in R^s. We define the s-variate B-spline $M\{x|x^0,\cdots,x^n)$, by requiring that the equation

$$\int_{R^s} f(x)M(x|x^0,\cdots,x^n)dx = n!\int_{S^n} f(\tau_0x^0+\cdots+\tau_nx^n)d\tau_1\cdots d\tau_n \tag{2.8}$$

holds for all $f \in L^1_{loc}(R^s)$, [12]. This definition requires $vol_s([K]) > 0$.

2.2. **The geometric interpretation of s-variate B-spline.**

Let $\sigma = [v^0,\cdots,v^n]$ be any n-simplex in R^nsuch that

$$v^i|_{R^s} = x^i, i = 0,\cdots,n. \tag{2.9}$$

We define the density function

$$M_\sigma(x) = vol_{n-s}(\{u\in\sigma : u|_{R^s} = x\}), x \in R^s. \tag{2.10}$$

An elementary calculation yields

$$\int_{R^s} f(x)M_\sigma(x)dx = n!vol_n\sigma\int_{S^n} f(\tau_0v^0+\cdots+\tau_nv^n|_{R^s})d\tau_1\cdots d\tau_n. \tag{2.11}$$

If we compare (2.8) and (2.11), we have

$$M(x|x^0,\cdots,x^n) = M_\sigma(x)/vol_n\sigma \tag{2.12}$$

As an example we consider the case $s = 1, n = 2$:
In Fig. 1, we have lifted x^0, x^1, x^2 into the plane forming the triangle T. The B-spline evaluated at t is

$$M(t|x^0,x^1,x^2) = \frac{\text{length } L}{\text{area } T}. \tag{2.13}$$

The right hand side of (2.12) is independent of σ, subject only to the condition (2.9). The formula (2.12) is certainly of no help for the calculation of M except for the case $n = s$ where we have

$$M(x|x^0, \cdots, x^s) = \mathcal{X}[x^0, \cdots, x^s](x)/vol_s([x^0, \cdots, x^s]). \tag{2.14}$$

The formulas (2.8) and (2.10) show that m is nonnegative and zero outside of $[x^0, \cdots, x^n]$. Since $M(x|x^0, \cdots, x^n)$ is independent of the ordering of the vectors $x^0, \cdots, x^n$,[13], we sometimes use the alternate notation $M(x|K)$ where $K = \{x^0, \cdots, x^n\}$.

2.3. **A recurrence relation for the multivariate B-spline.**

A practical computation of the B-spline is facilitated by the following recurrence relation. For $n \geq s + 1$ and $vol_s([x^0, \cdots, x^n]) > 0$, we have ([3],[12])

$$M(x|x^0, \cdots, x^n) = \frac{n}{n-s} \sum_{j=0}^{n} \lambda_j M(x|x^0, \cdots, x^{j-1}, x^{j+1}, \cdots, x^n) \tag{2.15}$$

whenever

$$x = \sum_{j=0}^{n} \lambda_j x^j, \sum_{j=0}^{n} \lambda_j = 1. \tag{2.16}$$

The λ_j's are not uniquely determined but one possible choice is as follows. For $x \in [x^0, \cdots, x^n]$, choose an $(s+1)$ affinely independent knots $x^{i_0}, \cdots, x^{i_s}$ so that $x \in [x^{i_0}, \cdots, x^{i_s}]$,(possible by Caratheodory theorem) and choose λ_j as the barycentric coordinates. That is,

$$\lambda_{i_j}(x|x^{i_0}, \cdots, x^{i_s}) = \frac{det \begin{pmatrix} x^{i_0}, & \cdots, & x^{i_{j-1}}, & x, & x^{i_{j+1}}, & \cdots, & x^{i_s} \\ 1, & \cdots, & 1, & 1, & 1, & \cdots, & 1 \end{pmatrix}}{det \begin{pmatrix} x^{i_0}, & \cdots, & x^{i_s} \\ 1, & \cdots, & 1 \end{pmatrix}}$$

where

$$det \begin{pmatrix} x^0, & \cdots, & x^s \\ 1, & \cdots, & 1 \end{pmatrix} = \begin{vmatrix} x_1^0, & \cdots, & x_1^s \\ \vdots & & \vdots \\ x_s^0, & \cdots, & x_s^s \\ 1, & \cdots, & 1 \end{vmatrix}$$

For $m \neq i_j$, we see $\lambda_m = 0$ so that (2.16) is satisfied. The univariate formula (2.3) corresponds to the choice $s = 1, i_0 = 0, i_s = n$.

2.4. **Continuity of B-spline.**

There is also a multivariate version of formula (2.5) expressing any directional derivative of a B-spline as a linear combination of lower-order B-splines. If $D_y = \sum_{i=1}^{s} y_i \frac{\partial}{\partial x_i}$, then one has , [13],

$$D_y M(x|x^0, \cdots, x^n) = \sum_{j=0}^{n} \mu_j M(x|x^0, \cdots, x^{j-1}, x^{j+1}, \cdots, x^n), \tag{2.17}$$

whenever

$$y = \sum_{j=0}^{n} \mu_j x_j, \sum_{j=0}^{n} \mu_j = 0. \tag{2.18}$$

As before we can pick any affinely independent knots $x^{i_0}, \cdots, x^{i_s}$ and set

$$\mu_m = \begin{cases} D_y \lambda_m(x|x^{i_0}, \cdots, x^{i_s}) & ; m = i_j, j = 0, \cdots, s, \\ 0 & ; \text{otherwise}, \end{cases}$$

where λ_m is as in (2.15) . It is easy to verify that the coefficients μ_m satisfy (2.18) so that the sum on the right hand side of (2.17) has at most $s+1$ nonzero terms. For $s = 1$, the choice $i_0 = 0, i_s = n$ gives the univariate formula (2.5) . There are a few noteworthy consequences of the relations (2.15) and (2.17). Both formulas readily show that $M(x|x^0, \cdots, x^n)$ is indeed a piecewise polynomial in each region which is bounded by but not intersected by the $(s-1)$ simplex by any s of its knots. Repeated applications of (2.5) show that

$$M(x|x^0, \cdots, x^n) \in C^{n-d}(R^s),$$

provided the convex hull of every subset of d knots taken from $\{x^0, \cdots, x^n\}$ has nonvanishing s-dimensional volume. C^k denotes the class of k times continuously differentiable functions in $R^s (k \geq 0)$ and C^{-1} the class of all piecewise continuous functions in R^s. In particular, if the knots are in general position, then the B-spline is in C^{n-s-1}.

Let us mention that in the extreme case where there are only $s+1$ distinct knots, say $x^j, j = 0, \cdots, s$, each repeated with multiplicities $m_j + 1$. $M(x|x^0, \cdots, x^n)$ is a polynomial on the simplex $[x^0, \cdots, x^n] = [x^0, \cdots, x^s]$. Moreover, one can show that it is given explicitly by

$$M(x|x^0, \cdots, x^n) = \frac{\mathcal{X}[x^0, \cdots, x^n](x)}{vol_s([x^0, \cdots, x^n])} \frac{(\lambda_0(x))^{m_0}}{m_0!} \cdots \frac{(\lambda_s(x))^{m_s}}{m_s!},$$

where

$$\lambda_j(x) = \lambda_j(x|x^0, \cdots, x^s), j = 0, \cdots, s,$$
$$x = \lambda_0(x)x^0 + \cdots + \lambda_s(x)x^s,$$
$$\lambda_0(x) + \cdots + \lambda_s(x) = 1.$$

3. A spline space

Let T be a triangulation of $\Omega \subset R^s$, [5]. That is , T is a collection of s-simplices such that the intersection of any two elements of T is either empty or a common lower dimensional simplex . Let Ω be any polyhedral set in R^s and suppose that $K_0 = \{x^{i,0} : i = 1, \cdots, N\}$ is any set of distinct vectors in Ω. Furthermore we let T be any triangulation of Ω so that K_0 is the set of vertices of the simplices in T. A typical simplex in T has the form $\rho_\alpha = [x^{\alpha_0,0}, \cdots, x^{\alpha_s,0}], 1 \leq \alpha_0 < \cdots < \alpha_s \leq N$ where $\alpha = (\alpha_0, \cdots, \alpha_s)$ and we index all such simplices by a set $J \in Z_+^{s+1}$, that is,

$$T = \{\rho_\alpha : \alpha \in J\}. \tag{3.1}$$

As an example,let Ω and K_0 be as follows:

$$K_0 = \{x^{i,0} : i = 1, \cdots, 6\}, J = \{(1,2,3), (2,3,5), (3,4,5), (4,5,6)\}.$$

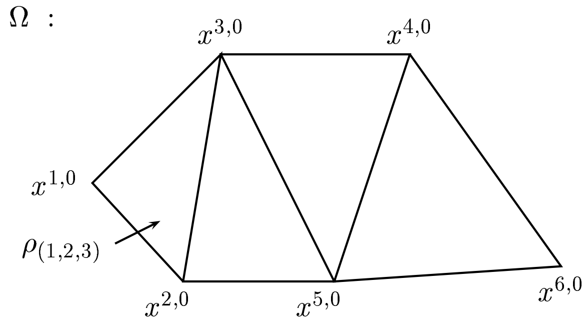


FIGURE 2

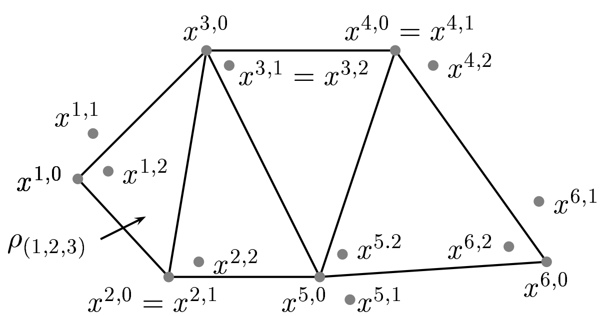


FIGURE 3

If $\rho(1,2,3)$ is the 2-simplex formed by $x^{1,0}, x^{2,0}, x^{3,0}$, say, then $T = U_{\alpha \in J}\rho_\alpha$.

In order to construct a spline space of degree k on Ω, we extend the list of vectors in K_0 to form $K_e = \{x^{i,j} : i = 1, \cdots, N, j = 0, \cdots, k\}$ by adding k additional vectors $x^{i,j}, j = 1, \cdots$,k corresponding to each $x^{i,0}$. We usually choose the vectors $x^{i,1}, \cdots, x^{i,k}$ as either being equal to or near $x^{i,0}$.Thus we might have, for example $s = 2$ and $k = 2$ an extended collection of vectors as in Fig. 3.

We will now describe a method of grouping the vectors in K_e into knot sets having cardinality $n + 1 = s + k + 1$. This will be done by forming groups of vectors from the extended set which corresponds to each simplex ρ_α in the triangulation T. For this purpose we define, for any s,k,

$$\begin{aligned}\Delta(s,k) &= \{Y = \{(i_0, m_0), \cdots, (i_n, m_n)\}, Y \subset \{0, \cdots, s\} \times \{0, \cdots, k\},\\ &\quad i_0 = m_0 = 0, (i_j, m_j) \le (i_{j+i}, m_{j+1}), j = 0, 1, \cdots, n-1\}.\end{aligned}$$

It is easy to show that

$$|\Delta(s,k)| = \binom{s+k}{s}.$$

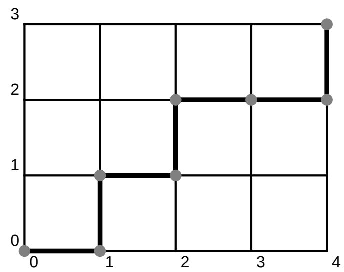


FIGURE 4

The set $\Delta(s,k)$ has the following obvious interpretation. Every set $Y \in \Delta(s,k)$ can be identified with a nondescending path along grid lines formed by the lattice points $(i,m), 0 \le i \le s, 0 \le m \le k$, which starts at $(0,0)$ and terminates at (s,k).

For instance, when $s = 4$ and $k = 3$ the dots in Fig. 4 indicate a typical set Y in $\Delta(4,3)$ Now with each $\beta = \{\beta_0, \cdots, \beta_s\} \in J$ we associate the local knot set configuration

$$K_\beta = \{K = \{x^{(\beta_l,m)} : (l,m) \in Y\} : Y \in \Delta(s,k)\}.$$

Referring to Fig. 3 where $s = k = 2$, we obtain

$$\Delta(2,2) =$$

$$\left\{\begin{matrix} \{(0.0),(1,0),(2,0),(2,1),(2,2)\}, & \{(0,0),(1,0),(1,1),(2,1),(2,2)\}, \\ \{(0.0),(1,0),(1,1),(1,2),(2,2)\}, & \{(0,0),(0,1),(1,1),(2,1),(2,2)\}, \\ \{(0.0),(0,1),(1,1),(1,2),(2,2)\}, & \{(0,0),(0,1),(0,2),(1,2),(2,2)\} \end{matrix}\right\}.$$

For $\beta = (2,3,5)$, for example, we obtain the local knot set configuration

$$K_\beta = \left\{\begin{matrix} \{x^{2,0},x^{3,0},x^{5,0},x^{5,1},x^{5,2}\}, & \{x^{2,0},x^{3,0},x^{3,1},x^{5,1},x^{5,2}\}, \\ \{x^{2,0},x^{3,0},x^{3,1},x^{3,2},x^{5,2}\}, & \{x^{2,0},x^{2,1},x^{3,1},x^{5,1},x^{5,2}\}, \\ \{x^{2,0},x^{2,1},x^{3,1},x^{3,2},x^{5,2}\}, & \{x^{2,0},x^{2,1},x^{2,2},x^{3,2},x^{5,2}\} \end{matrix}\right\}.$$

The global knot set configurations are then obtained by taking the union of all the local knot set configurations

$$GK = \bigcup_{\beta \in J} K_\beta. \tag{3.2}$$

Thus we have $|GK| = \binom{s+k}{s}|J|$. We note that GK depends only on the numbering of the vectors in GK and on the triangulation T. A spline space is given by $S(GK) = \text{span}\{M(x|K) : K \in GK\}$.

4. THE LEAST SQUARES APPROXIMATION BY MULTIVARIATE B-SPLINES

The following information is assumed to be prescribed:

a polyhedral set Ω;

global knot set configuration $GK = \{K_i : i = 1, \cdots, q\}, q = |GK|$;

data points $(x^i, f^i), i = 1, \cdots, m$, with $x_i \in \Omega$;

positive weights $w_1, \cdots, w_m$.
We want to compute the spline

$$s(x) = \sum_{i=1}^{q} c_i M(x|K_i) \tag{4.1}$$

of degree 3 in R^2 which best fits the data $(x^i, f^i), i = 1, \cdots, m$ in the least squares sense. That is, we want to minimize the weighted residual sum of the squares of the errors

$$E = E(c_1, \cdots, c_q) = \sum_{i=1}^{m} w_i^2 (f^i - \sum_{j=1}^{q} c_j B_j(x^i))^2, \tag{4.2}$$

where $B_j(x) = M(x|K_j)$. Thus $c = (c_1, \cdots, c_q)$ must satisfy the normal equations, $\frac{\partial E(c)}{\partial c} = 0$, or,

$$-2 \sum_{i=1}^{m} w_i^2 (f^i - \sum_{j=1}^{q} c_j B_j(x^i)) B_k(x^i) = 0, k = 1, \cdots, q. \tag{4.3}$$

We rewrite the normal equations (4.3) in the form

$$\sum_{j=1}^{q} c_j \sum_{i=1}^{m} w_i^2 B_j(x^i) B_k(x^i) = \sum_{i=1}^{m} w_i^2 f^i B_k(x^i), k = 1, \cdots, q, \tag{4.4}$$

which can be written in the matrix form

$$Ac = b \tag{4.5}$$

where A is the $q \times q$ matrix whose (j,k) element is $\sum_{i=1}^{m} w_i^2 B_j(x^i) B^k(x^i)$, and c and b are column vectors with elements c_j and $\sum_{i=1}^{m} w_i^2 f^i B_j(x^i)$, respectively. The matrix A turns out to be large sparse, symmetric matrix.

5. Bases

In our B-spline approximation, we use the normalized B-spline bases

$$N(x|x^0, \cdots, x^n) = \frac{(n-s)!s!}{n!} M(x|x^0, \cdots, x^n).$$

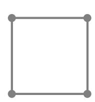
Then the recurrence relation (2.15) takes the form

$$N(x|x^0, \cdots, x^n) = \sum_{j=0}^{n} \lambda_j N(x|x^0, \cdots, x^{j-1}, x^{j+1}, \cdots, x^n),$$

Where λ_j's are in (2.16). Using this recurrence relation we compute $N(x|x^0, \cdots x^n)$ for fourteen different knot configurations of points in the (x, y)-plane. The resulting discrete representations of the B-spline basis functions are then graphically displayed in Figs. 6-8. In our polynomial approximation, we use as bases the homogeneous monomials given in the array of Pascal triangle as in Fig. 5.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & x & & y & & & \\
 & & x^2 & & xy & & y^2 & & \\
 & & & \cdot & \cdot & \cdot & & & \\
x^n & & x^{n-1}y & \cdot & \cdot & \cdot & xy^{n-1} & & y^n
\end{array}
$$

FIGURE 5



$(0,0)(1,0)(1,1)(0,1)$

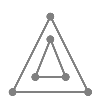
$s = 2, n = 3$; degree 1, $C(\mathbf{R}^2)$

FIGURE 6

6. NUMERICAL RESULTS

6.1. Test functions for approximating schemes.

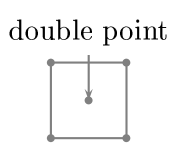
In our numerical results, we use as test functions the functions given in (A)-(G).
(A) f(x,y)=$\frac{1}{1+100(x^2+y^2)}$
(B) f(x,y)=$\cos(100x^2y)$
(C) f(x,y)=$\exp(5xy)$
(D) f(x,y)=$3|x+y|$
(E) f(x,y)=$\sin(16\pi xy)$
(F) f(x,y)=$\cos(50x^2y)$
(G) f(x,y)=$\sin(8\pi xy)$
We use the following schemes (I)-(V) for numerical tests. The weights w_i are all set to



$(0,0)(1,0)(1/2,1)(1/4,1/4)(3/4,1/4)(1/2,3/4)$

$s=2, n=5$; degree 3, $C^2(\mathbf{R}^2)$

FIGURE 7



$(0,0)(1,0)(1,1)(0,1)(1/2,1/2)(1/2,1/2)$

$s=2, n=5$; degree 3, $C^1(\mathbf{R}^2)$

FIGURE 8

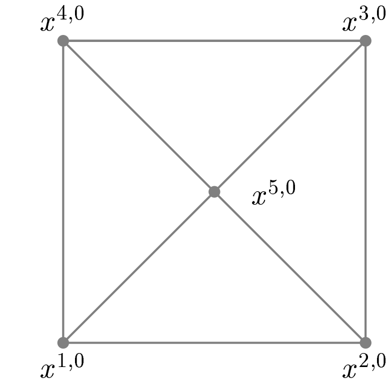


FIGURE 9

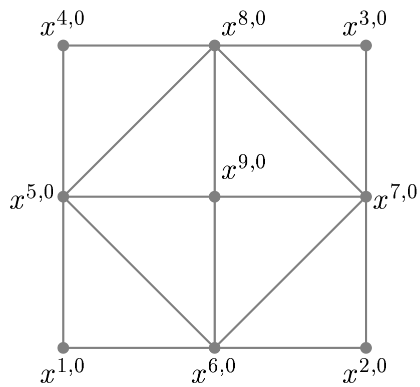


FIGURE 10

unity throughout all the schemes. The only difference of the schemes (I) and (II) are the knot positions. In the scheme (III), the domain is 4 times larger than in (I) or (II). The schemes (IV) and (V) are for the polynomial approximations.

I. Polyhedral set Ω_1 is as in Fig. 9.

The number of basis elements is $q = 40$. The regular 15×15 grids of calculation points (x^i, f^i) on $[-.25, .24] \times [-.25, .24]$ were chosen as data points. The knot positions $x^{i,j}$ are $x^{1,0} = (-.25, -.25), x^{2,0} = (.25, -.25), x^{3,0} = (.25, .25), x^{4,0} = (-.25, .25), x^{5,0} = (0, 0)$ and $x^{i,0} = \cdots = x^{i,3}, i = 1, \cdots, 5$.

II. Polyhedral set Ω_2 is as in Fig. 10. The number of basis elements is $q = 80$. Data points are the same as in (I). The positions of knots $x^{i,j}$ are as in Table 1.

III. Polyhedral set Ω_3 is as in Fig. 11. The number of basis elements is $q = 320$. Regular 29×29 grids of calculations (x^i, f^i) on $[-.5, .48] \times [-.5, .48]$ were chosen as data points. The knot positions are

TABLE 1

(i,j)	0	1	2	3
1	(-0.26,-0.26)	(-0.30,-0.26)	(-0.26,-0.30)	(-0.26,-0.25)
2	(0.26,-0.26)	(0.30,-0.27)	(0.26,-0.30)	(0.27,-0.25)
3	(0.26,0.26)	(0.30,0.27)	(0.25,0.30)	(0.27,0.25)
4	(-0.26,0.26)	(-0.30,0.25)	(-0.26,0.30)	(-0.27,0.26)
5	(-0.26,0.00)	(-0.26,0.00)	(-0.26,0.00)	(-0.26,0.00)
6	(0.00,-0.26)	(0.00,-0.26)	(0.00,-0.26)	(0.00,-0.26)
7	(0.26,0.00)	(0.26,0.00)	(0.26,0.00)	(0.26,0.00)
8	(0.00,0.26)	(0.00,0.26)	(0.00,0.26)	(0.00,0.26)
9	(0.00,0.00)	(0.00,0.0.00)	(0.00,0.00)	(0.00,0.00)

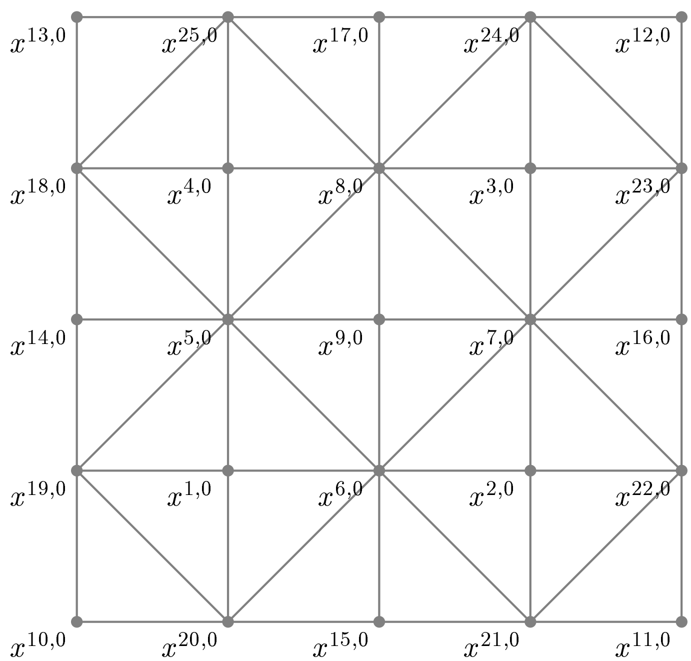


FIGURE 11

$$
\begin{array}{lll}
x^{1,0} = (-.25, -.25), & x^{2,0} = (.25, -.25), & x^{3,0} = (.25, .25), \\
x^{4,0} = (-.25, .25), & x^{5,0} = (-.25, 0), & x^{6,0} = (0, -.25), \\
x^{7,0} = (.25, 0), & x^{8,0} = (0, .25), & x^{9,0} = (0, 0), \\
x^{10,0} = (-.502, -.502), & x^{11,0} = (.502, -.502), & x^{12,0} = (.502, .502), \\
x^{13,0} = (-.502, .502), & x^{14,0} = (-.502, 0), & x^{15,0} = (0, -.502), \\
x^{16,0} = (.502, 0), & x^{17,0} = (0, .502), & x^{18,0} = (-.502, .25), \\
x^{19,0} = (-.502, -.25), & x^{20,0} = (-.25, -.502), & x^{21,0} = (.25, -.502), \\
x^{22,0} = (.502, -.25), & x^{23,0} = (.502, .25), & x^{24,0} = (.25, .502), \\
x^{25,0} = (-.25, .502) & &
\end{array}
$$

and $x^{i,0} = \cdots = x^{i,3}, i = 1 \cdots 25$.

IV. Data points are the same as in (I). Polynomial basis of total degree 8 is used. The number of basis elements is 36.

V. Data points are the same as in (III). Polynomial basis of total degree 8 is used. The number of basis elements are 36.

6.2. **Graphical comparisons.**

We compare the original surface with the approximated fittings. We approximate the surfaces (A)-(G) by the approximation schemes (I)-(V). The Figs. 12-17 are graphical comparisons.

6.3. **Errors.**

We compare the errors among the approximation schemes. In Tables 2-3, the average absolute errors and the average relative errors are compared for functions (A) - (G) in the approximation schemes (I),(II) and (IV). Tables 4-5 compare the errors in schemes (III) and (V).

TABLE 2. Average absolute error

Functions	Scheme(I)	Scheme(II)	Scheme(IV)
$\frac{1}{1+100(x^2+y^2)}$	1.0918e-2	4.6651e-3	2.5118e-2
$\cos(100x^2y)$	1.7329e-2	1.4776e-3	2.9632e-3
$\exp(5xy)$	1.7657e-4	5.9417e-5	2.5023e-6
$3\|x+y\|$	2.5717e-6	1.3480e-2	2.7416e-2
$\sin(16\pi xy)$	3.7729e-2	5.1286e-3	1.6429e-2

TABLE 3. Average relative error

Functions	Scheme(I)	Scheme(II)	Scheme(IV)
$\frac{1}{1+100(x^2+y^2)}$	4.9916e-2	1.5479e-2	1.2248e-1
$\cos(100x^2y)$	8.7580e-2	3.3665e-3	1.1063e-2
$\exp(5xy)$	1.7188e-4	5.8826e-5	2.5014e-6
$3\|x+y\|$	1.2185e-5	1.5664e-1	2.3931e-1
$\sin(16\pi xy)$	2.3976e-1	6.7305e-2	2.1213e-1

TABLE 4. Average absolute error

Functions	Scheme(III)	Scheme(V)
$\frac{1}{1+100(x^2+y^2)}$	1.0696e-3	3.4757e-2
$\cos(100x^2y)$	7.3530e-3	2.1149e-0
$\exp(5xy)$	7.9110e-5	5.7887e-4
$3\|x+y\|$	6.4370e-3	5.4138e-2
$\sin(16\pi xy)$	3.1792e-3	1.3312e-1

TABLE 5. Average relative error

Functions	Scheme(III)	Scheme(V)
$\frac{1}{1+100(x^2+y^2)}$	4.1144e-3	4.6234e-1
$\cos(100x^2y)$	2.5439e-2	3.5786e-1
$\exp(5xy)$	6.3108e-5	5.7966e-4
$3\|x+y\|$	5.8683e-2	2.7154e-1
$\sin(16\pi xy)$	4.1738e-2	4.4043e-1

FIGURE 12. (1) Test function (A) $f(x,y) = \frac{1}{1+100(x^2+y^2)}$ (original surface)

FIGURE 13. Approximated surface by scheme (III) (spline approximation)

6.4. **Analysis.**

The Tables 3, 5 and the corresponding graphical representations show that the B-spline approximation is better than the polynomial approximation for the surfaces

FIGURE 14. Approximated surface by scheme (V) (polynomial approximation)

FIGURE 15. (2) Test function (C) $f(x, y) = \exp(5xy)$ (original surface)

(A),(B),(E),(F),(G) which vary rapidly. Especially, for the surface (A) which vary very rapidly, the spline approximation is much better than the polynomial approximation . For the surface (C) which vary slowly, the polynomial approximation is as good as the spline approximation as seen in Tables 3, 5 and Figs. 15-17. In the larger domains, the

FIGURE 16. Approximated surface by scheme (III) (spline approximation)

FIGURE 17. Approximated surface by scheme (V) (polynomial approximation)

B-spline approximation is better than the polynomial approximation for all the cases we consider. See Table 5. For the surface (D), the scheme (I) is better than (II) as seen in Table 3. This is because we divided the region according to the wedge of the surface (D) in the case (I).

6.5. **Conclusions.**

The multivariate B-spline approximation needed more work than that of the polynomial approximation. It approximated most of the surfaces we considered much better than the polynomial approximation. Especially, for the surface which vary rapidly, the multivariate B-spline is much better than the polynomial one. If we know some properties of the surface, we can take those into account in taking the knot positions to get better approximation . This is impossible in polynomial approximation. When we want to approximate a surface globally in a larger domain, the multivariate B-spline seems to be very good in getting an approximated surface. In order to fit a surface with a discrete data , the multivariate B-spline would give a good approximated surface.

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