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Fully discrete Galerkin method for a unidimensional single-phase nonlinear Stefan problem with Neumann boundary conditions

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Abstract

In this paper we analyze the error estimates for a single-phase nonlinear Stefan problem with Neumann boundary conditions. We apply the modified Crank-Nicolson method to get the optimal order of error estimates in the temporal direction.

1. Introduction

In this paper we consider the finite element fully discrete approximation to the following single-phase nonlinear Stefan problem.

Find a pair $\{(U,S) : U = U(y,\tau) \text{ and } S = S(\tau)\}$ such that U satisfies

$$U_\tau - (a(U)U_y)_y = 0 \quad \text{in} \quad \Omega(\tau) \times (0, T_0], \tag{1.1}$$

with initial and boundary conditions

$$U(y,0) = g(y) \qquad \text{for} \quad y \in I, \tag{1.2}$$

$$U_y(0,\tau) = U(S(\tau),\tau) = 0 \quad \text{for} \quad 0 < \tau \le T_0, \tag{1.3}$$

and further, on the free boundary, S satisfies

$$S_\tau = -a(U)U_y \qquad \text{for} \quad 0 < \tau \le T_0 \tag{1.4}$$

with $S(0) = 1$, where $\Omega(\tau) = \{y \mid 0 < y < S(\tau)\}$ for each $\tau \in (0, T_0]$ and $I = (0,1)$. For a single-phase linear Stefan problem, the study of semidiscrete finite element error analysis was initiated with the fixed domain method by Nitsche[9,10] using the fixing domain method. Das and Pani[3] have extended the error-analysis to nonlinear problem and derived optimal estimates in H^1 and H^2 norms for semidiscrete Galerkin approximations . And when the temperature was given at the fixed boundary instead of the

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flux condition, Das and Pani[3,4] obtained error estimates for a semidiscrete Galerkin approximation. Also error estimates for fully discrete Galerkin approximation, depending on the backward Euler method in time, were derived in [12]. Lee and Lee[7] adopted the modified Crank-Nicolson method to improve the rate of convergence in the temporal direction for a single-phase nonlinear Stefan problem with Dirichlet boundary condition. Lee, Ohm and Shin[8] analyzed the optimal convergence of semi-discrete approximation in L_2 norm for a single-phase nonlinear Stefan problem with Neumann boundary condition.

In this paper we consider the optimal convergence of fully discrete approximation for a single-phase nonlinear Stefan problem in one space dimension with Neumann boundary condition and we achieve the convergence of order 2 in the temporal direction.

For simplicity, we suppress τ in $\Omega(\tau)$ and write $\Omega(\tau)$ as Ω only.

For an integer $m \geq 0$, and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ will denote the usual Sobolev space of measurable functions which, together with their distributional derivatives of order up to m, are in L^p. For $\Omega = I$ and $p = 2$, we shall use the symbol H^m in place of $W^{m,2}(I)$ with norm $\|\cdot\|_m$.

Let $Y(\tau)$ be a Banach space, for each fixed $\tau \geq 0$ with norm $\|\cdot\|_{Y(\tau)}$. The following notation is used :

$$\begin{aligned} \|v\|_{L^p(0,T;Y(\tau))} &= (\int_0^T \|v(\tau)\|^p_{Y(\tau)} d\tau)^{\frac{1}{p}}, \quad \text{for} \quad 1 \leq p < \infty \\ \|v\|_{L^\infty(0,T;Y(\tau))} &= \sup_{0\leq\tau\leq T} \|v(\tau)\|_{Y(\tau)}, \end{aligned}$$

where $Y(\tau)$ is $W^{m,p}(\Omega)$.

Throughout this paper, we assume the following regularity conditions on $\{U, S\}$:

Condition I :

(i) The pair $\{U, S\}$ is the unique smooth solution to (1.1)-(1.4) with $S(\tau) \geq \nu > 0$ for all $\tau \in [0, T_0]$.

(ii) The function $a(\cdot)$ belongs to $C^4(\mathbf{R})$ and has bounded derivatives up to order 4, bounded by a common constant $\tilde{K}_1$, further, there exists $\tilde{\alpha} > 0$ such that $a(w) \geq \tilde{\alpha}$ for all $w \in \mathbf{R}$.

(iii) The initial function g is sufficiently smooth nonnegative, and satisfies the compatibility conditions $g(0) = g(1) = 0$.

Condition II : For $r \geq 1$

$$\begin{aligned} U &\in W^{3,\infty}(0, T_0; H^{r+1}(\Omega)) \\ S &\in W^{3,2}(0, T_0). \end{aligned}$$

Let $\tilde{K_2}$ be a bound for $\{U, S\}$ in the spaces appeared in condition II.

Throughout this paper, we frequently use the following inequalities for error estimates.

Young's inequality : For nonnegative real numbers a and b and a positive number ϵ,

$$ab \leq \frac{(\epsilon a)^p}{p} + \frac{1}{q}(\frac{b}{\epsilon})^q \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1$$

with $1 \leq p \leq \infty$.
Sobolev imbedding inequality: For $\phi \in H^2$,

$$\sup_{0 \leq x \leq 1} (|\phi(x)| + |\phi_x(x)|) \leq \|\phi\|_2,$$

$$|\phi_x(1)| \leq \|\phi_x\| + \sqrt{2}\|\phi_x\|^{\frac{1}{2}}\|\phi_{xx}\|^{\frac{1}{2}}.$$

Poincare's inequality: For $\phi \in H_0^1$, $\quad \|\phi\| \leq \|\phi_x\|$.

2. Weak formulation and Galerkin approximations

By the application of the following Landau transformations

$$x = yS^{-1}(\tau) \quad \text{and} \quad t(\tau) = \int_0^\tau S^{-2}(\tau')d\tau', \tag{2.1}$$

the given problem (1.1)-(1.4) can be transformed into the following problem:
Find a pair $\{(u,s); u(x,t) \equiv U(y,\tau)$ and $s(t) \equiv S(\tau)\}$ such that

$$u_t - (a(u)u_x)_x = -a(0)u_x(1)xu_x \quad \text{for} \quad (x,t) \in I \times (0,T], \tag{2.2}$$

with

$$u(x,0) = g(x) \quad \text{for} \quad x \in I \tag{2.3}$$

$$u_x(0,t) = u(1,t) = 0 \quad \text{for} \quad 0 < t \leq T \tag{2.4}$$

$$\frac{ds}{dt} = -a(0)u_x(1)s \quad \text{for} \quad 0 < t \leq T \tag{2.5}$$

with $s(0) = 1$. Here, $t = T$ corresponds to $\tau = T_0$ and $u_x(1) = (\frac{\partial u}{\partial x})(1,t)$.

Note that the regularity properties in condition II for $\{U,S\}$ are transferred to $\{u,s\}$ and call these conditions II with the bounds K_2.
And also from (2.1) we can get the following:

$$\begin{aligned} \frac{d\tau}{dt} &= s^2(t) \quad \text{for} \quad 0 < t \leq T \\ \tau(0) &= 0. \end{aligned}$$

Now we introduce a Sobolev space $H_0^2 = \{v \in H^2; v_x(0) = v(1) = 0\}$.

Multiplying both sides of (2.2) by v_{xx} and integrating by parts the first term of the left-hand side with respect to x, we get

$$(u_{tx}, v_x) + ((a(u)u_x)_x, v_{xx}) = a(0)u_x(1)(xu_x, v_{xx}) \quad \text{for} \quad v \in H_0^2. \tag{2.6}$$

To get Galerkin approximation of u in the finite element space, we introduce a family of finite dimensional subspaces $\{S_h^0\}$ of H_0^2, satisfying the following approximation property and inverse property:

There is a constant K_0 independent of h such that

$$\inf_{\chi \in S_h^0} \|v - \chi\|_j \le K_0 h^{m-j} \|v\|_m, \tag{2.7}$$

for any $v \in H^m \cap H_0^2$ for $j = 0, 1, 2$ and $2 \le m \le r+1$. And

$$\|\chi\|_2 \le K_0 h^{-1} \|\chi\|_1, \quad \text{for} \;\; \chi \in S_h^0. \tag{2.8}$$

Now we define Galerkin approximation as follows. Find $u^h : [0, T] \to S_h^0$ such that

$$(u_{tx}^h, \chi_x) + ((a(u^h)u_x^h)_x, \chi_{xx}) = a(0)u_x^h(1)(xu_x^h, \chi_{xx}), \quad \forall \chi \in S_h^0, \tag{2.9}$$

with

$$u^h(x, 0) = Q_h g(x)$$

where Q_h is an appropriate projection onto S_h^0 to be defined later in section 4. Moreover Galerkin approximations s_h and τ_h of s and τ, respectively are defined by

$$\frac{ds_h}{dt} = -a(0)u_x^h(1)s_h \quad \text{for} \quad t \ge 0 \tag{2.10}$$

with

$$s_h(0) = 1$$

and

$$\frac{d\tau_h}{dt} = s_h^2(t) \quad \text{for} \quad t \ge 0 \tag{2.11}$$

with

$$\tau_h(0) = 0.$$

Next we define a fully discrete approximation. To avoid having a nonlinear term, we adopt the modified Crank-Nicolson method which yields second-order accuracy in time. Let $k = \frac{T}{N}$ be the step size in time, and $t^n = nk, \quad n = 0, 1, 2, \cdots, N$. The modified Galerkin Crank-Nicolson approximation $\{Z^m\}_{m=2}^N$ is defined as follows :

$$(d_t Z_x^n, \chi_x) + ((a(\widehat{Z^n})\overline{Z_x^n})_x, \chi_{xx}) = a(0)\widehat{Z_x^n}(1)(x\overline{Z_x^n}, \chi_{xx}), \tag{2.12}$$

$$\forall \chi \in S_h^0, \quad 1 \le n \le N-1,$$

where $d_t Z_x^n = \frac{Z_x^{n+1} - Z_x^n}{k}$, $\quad \widehat{Z^n} = \frac{3}{2}Z^n - \frac{1}{2}Z^{n-1}$, and $\overline{Z_x^n} = \frac{Z_x^{n+1} + Z_x^n}{2}$.
Now 2 initial approximations Z^0, and Z^1, needed to apply (2.12), are defined in the following :

$$Z^0(x) = Q_h g(x) \quad \text{for} \quad 0 < x < 1 \tag{2.13}$$

$$(d_t Z_x^0, \chi_x) + ((a(\overline{Z^0})\overline{Z_x^0})_x, \chi_{xx}) = a(0)\overline{Z_x^0}(1)(x\overline{Z_x^0}, \chi_{xx}). \tag{2.14}$$

The approximation W^n of $s(t^n)$ is defined by

$$d_t W^n = -a(0) Z_x^{n+1}(1)\overline{W^n} \quad \text{for} \quad 0 \le n \le N-1, \tag{2.15}$$

$$W^0 = 1.$$

And also the approximation τ_h^n of $\tau(t^n)$ is

$$d_t \tau_h^n = (\overline{W^n})^2, \tag{2.16}$$

$$\tau_h^0 = 0.$$

3. Auxiliary projection and related estimates

For $u, v, w \in H_2^0(I)$, we define a trilinear form

$$A(u; v, w) = ((a(u)v_x + a_u(u)u_x v)_x, w_{xx}) - a(0)u_x(1)(xv_x, w_{xx}). \tag{3.1}$$

We can prove that

$$|A(u; v, w)| \le K_3 \|v\|_2 \|w\|_2 \tag{3.2}$$

$$A(u; v, v) \ge \alpha \|v\|_2^2 - \rho \|v\|_1^2 \tag{3.3}$$

for u, v and $w \in H_0^2$ where K_3, α, and ρ are constants depending on $\|u\|_2$ only.

Let

$$A_\rho(u; v, w) = A(u; v, w) + \rho(v_x, w_x). \tag{3.4}$$

Let $\tilde{u} \in S_h^0$ be an auxiliary projection of u with respect to the form A_ρ :

$$A_\rho(u; u - \tilde{u}, \chi) = 0 \quad \forall \ \chi \in S_h^0. \tag{3.5}$$

Theorem 3.1.[3] For sufficiently small h and a given $u \in H^2 \cap H_0^1$, there exists a unique solution $\tilde{u} \in S_h^0$ to (3.5)

Let $\eta = u - \tilde{u}$. Then we obtain the following estimates for η and η_t whose proofs are given in [8].

Theorem 3.2. For $t \in [0, T]$, there exists a constant $K_4 = K_4(K_0, K_1, K_2, K_3, \alpha, \rho)$ such that

$$\begin{aligned} \|\eta_t\|_j + \|\eta\|_j &\leq K_4 h^{m-j} \|u\|_m, \\ |\eta_x(1)| &\leq K_4 h^{2(m-2)} \|u\|_m, \end{aligned}$$

hold for $j = 0, 1, 2$ and $2 \leq m \leq r + 1$.

As a corollary to Theorem 3.2, there exists K_5 such that

$$\|\tilde{u}\|_{L^\infty(H^2)} + \|\tilde{u}_t\|_{L^\infty(H^2)} \leq K_5. \tag{3.6}$$

4. Error estimates for the fully discrete approximation

To get the error estimates for $u^n - Z^n$ and $s^n - W^n$, we introduce $\eta^n = u^n - \tilde{u}^n$, $\zeta^n = Z^n - \tilde{u}^n$ and $e^n = u^n - Z^n$. Further let $Z^0 = \tilde{u}(x, 0)$, i.e.,

$$A(g; g - Q_h g, \chi) = 0 \quad \text{for} \quad \chi \in S_h^0.$$

We assume that there exists a constant K_6 such that

$$\|Z^n\|_{W^{1,\infty}} \leq K_6 \quad \text{for} \quad n = 0, 1, 2, \cdots, N. \tag{4.1}$$

The ϵ appearing in the theorems in this section is an arbitrary positive real number. Especially, we assume that ϵ is sufficiently small whenever it is needed.

In the following theorem we estimate the error bound for ζ^1 .

Theorem 4.1. There exist $K_7 = K_7(K_1, K_2, K_4, K_5, K_6, \rho)$ and h_0 such that for $h \leq h_0, k = O(h), 4 \leq m \leq r + 1$ and $\tilde{\beta} > 0$,

$$\|\zeta_x^1\|^2 + \tilde{\beta} k \|\zeta_{xx}^1\|^2 \leq K_7(K_1, K_2, K_4, K_5, K_6, \rho) k \{h^{2m} + k^4\} \tag{4.2}$$

holds.

Proof. From (3.5) and (2.6) with $v = \chi$ we have,

(4.3)

$$\begin{aligned} (d_t \tilde{u}_x^0, \chi_x) + ((a(\tilde{u}^{\frac{1}{2}}) \tilde{u}_x^{\frac{1}{2}})_x, \chi_{xx}) &= -(d_t \eta_x^0, \chi_x) - ((\frac{\partial^2 u}{\partial t \partial x})^{\frac{1}{2}} - d_t u_x^0, \chi_x) \\ &\quad + a(0) u_x^{\frac{1}{2}}(1)(x \tilde{u}_x^{\frac{1}{2}}, \chi_{xx}) + ((a_u(u^{\frac{1}{2}}) \eta^{\frac{1}{2}} u_x^{\frac{1}{2}})_x, \chi_{xx}) \\ &\quad - (([a(u^{\frac{1}{2}}) - a(\tilde{u}^{\frac{1}{2}})] \tilde{u}_x^{\frac{1}{2}})_x, \chi_{xx}) + \rho(\eta_x^{\frac{1}{2}}, \chi_x) \end{aligned}$$

The fifth term on the right-hand side of (4.3) can be rewritten as,

(4.4)

$$
\begin{aligned}
&(([a(u^{\frac{1}{2}})-a(\tilde{u}^{\frac{1}{2}})]\eta_x^{\frac{1}{2}})_x-([a(u^{\frac{1}{2}})-a(\tilde{u}^{\frac{1}{2}})]u_x^{\frac{1}{2}})_x,\chi_{xx})\\
&=((\tilde{a}_u\eta^{\frac{1}{2}}\eta_x^{\frac{1}{2}})_x,\chi_{xx})-([a_u(u^{\frac{1}{2}})\eta^{\frac{1}{2}}u_x^{\frac{1}{2}}]_x,\chi_{xx})+([\tilde{a}_{uu}(\eta^2)^{\frac{1}{2}}u_x^{\frac{1}{2}}]_x,\chi_{xx})
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{a}_u &= \int_0^1 \frac{\partial a}{\partial u}(u^{\frac{1}{2}}-\xi\eta^{\frac{1}{2}})d\xi,\\
\tilde{a}_{uu} &= \int_0^1 \frac{\partial^2 a}{\partial u^2}(u^{\frac{1}{2}}-\xi\eta^{\frac{1}{2}})(1-\xi)d\xi.
\end{aligned}
$$

Substituting (4.4) in (4.3) and subtracting (4.3) from (2.14), we have
(4.5)

$$
\begin{aligned}
(d_t\zeta_x^0,\chi_x)+((a(\overline{Z^0})\overline{\zeta_x^0})_x,\chi_{xx}) &= -(d_t\eta^0+u_t^{\frac{1}{2}}-d_tu^0,\chi_{xx})\\
&\quad +a(0)[\overline{Z_x^0}(1)(x\overline{Z_x^0},\chi_{xx})-u_x^{\frac{1}{2}}(1)(x\tilde{u}_x^{\frac{1}{2}},\chi_{xx})]\\
&\quad -((a(\overline{Z^0})\overline{\tilde{u}_x^0})_x,\chi_{xx})+((a(\tilde{u}^{\frac{1}{2}})\tilde{u}_x^{\frac{1}{2}})_x,\chi_{xx})\\
&\quad -((\tilde{a}_u\eta^{\frac{1}{2}}\eta_x^{\frac{1}{2}})_x,\chi_{xx})-([\tilde{a}_{uu}(\eta^2)^{\frac{1}{2}}u_x^{\frac{1}{2}}]_x,\chi_{xx})\\
&\quad +\rho(\eta_x^{\frac{1}{2}},\chi_x)\\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Now we need to find the estimates for I_1, I_2, I_3, I_4 and I_5. To get the estimates, we substitute $\chi=\zeta^1$ in (4.5). First we get the estimation for I_1 in the following

$$
\begin{aligned}
|I_1| &= |(d_t\eta^0+u_t^{\frac{1}{2}}-d_tu^0,\zeta_{xx}^1)|\\
&\le \frac{1}{2\epsilon}(\|d_t\eta^0\|^2+\|d_tu^0-u_t^{\frac{1}{2}}\|^2)+\epsilon\|\zeta_{xx}^1\|^2.
\end{aligned}
$$

To get an estimation for I_2, we separate I_2 into four terms in the following way

$$
\begin{aligned}
I_2 &= a(0)[\overline{Z_x^0}(1)(x\overline{\zeta_x^0},\zeta_{xx}^1)\\
&\quad +(\overline{\zeta_x^0}(1)-\overline{\eta_x^0}(1))(x\overline{\tilde{u}_x^0},\zeta_{xx}^1)\\
&\quad +\overline{u_x^0}(1)(x\overline{\tilde{u}_x^0},\zeta_{xx}^1)-u_x^{\frac{1}{2}}(1)(x\overline{\tilde{u}_x^0},\zeta_{xx}^1)\\
&\quad +u_x^{\frac{1}{2}}(1)(x(\overline{\tilde{u}_x^0}-\tilde{u}_x^{\frac{1}{2}}),\zeta_{xx}^1)]\\
&\equiv I_{21}+I_{22}+I_{23}+I_{24}.
\end{aligned}
$$

Now we will estimate $I_{2i}, 1\le i\le 4$. Applying condition I, (4.1) and Young's inequality to the term I_{21}, we have

$$
|I_{21}|\le K(\epsilon,K_1,K_6)(\|\zeta^0\|_1^2+\|\zeta^1\|_1^2)+\epsilon\|\zeta_{xx}^1\|^2.
$$

And using condition I, the Sobolev inequality, (3.6) and Young's inequality to I_{22}, we get

$$|I_{22}| \leq K(\epsilon, K_1, K_5)(\|\overline{\zeta_x^0}\|^2 + |\overline{\eta_x^0}(1)|^2) + \frac{1}{4}\epsilon^2\|\overline{\zeta_{xx}^0}\|^2 + \epsilon\|\zeta_{xx}^1\|^2.$$

To estimate I_{23}, we consider

$$\overline{u_x^0}(1) - u_x^{\frac{1}{2}}(1) = \frac{1}{4}k^2 u_{xtt}^0(\theta_1) - \frac{1}{8}k^2 u_{xtt}^0(\theta_2).$$

Thus we get

$$|I_{23}| \leq K(\epsilon, K_1, K_2, K_5)k^4 + \epsilon\|\zeta_{xx}^1\|^2.$$

Similarly, we obtain

$$|I_{24}| \leq K(\epsilon, K_1, K_2, K_5)k^4 + \epsilon\|\zeta_{xx}^1\|^2.$$

Therefore, we have

$$\begin{aligned} |I_2| \quad \leq \quad & K(\epsilon, K_1, K_2, K_5, K_6)\{\|\zeta^0\|_1^2 + \|\zeta^1\|_1^2 + \|\overline{\zeta_x^0}\|^2 + |\overline{\eta_x^0}(1)|^2 + k^4\} \\ & +4\epsilon\|\zeta_{xx}^1\|^2 + \frac{1}{4}\epsilon^2\|\overline{\zeta_{xx}^0}\|^2. \end{aligned}$$

To estimate I_3, we can rewrite I_3 as follows:

$$\begin{aligned} I_3 \quad = \quad & ((a(\tilde{u}^{\frac{1}{2}})\tilde{u}_x^{\frac{1}{2}} - a(\overline{\tilde{u}^0})\tilde{u}_x^{\frac{1}{2}})_x, \zeta_{xx}^1) \\ & +((a(\overline{\tilde{u}^0})\tilde{u}_x^{\frac{1}{2}} - a(\overline{\tilde{u}^0})\overline{\tilde{u}_x^0})_x, \zeta_{xx}^1) \\ & +((a(\overline{\tilde{u}^0})\overline{\tilde{u}_x^0} - a(\overline{Z^0})\overline{\tilde{u}_x^0})_x, \zeta_{xx}^1) \\ \equiv \quad & I_{31} + I_{32} + I_{33}. \end{aligned}$$

Using (3.6), condition I and Young's inequality, we obtain

$$\begin{aligned} |I_{31}| \quad & \leq \quad K(K_1, K_5, \epsilon)k^4 + 3\epsilon\|\zeta_{xx}^1\|^2, \\ |I_{32}| \quad & \leq \quad K(K_1, K_5, \epsilon)k^4 + 2\epsilon\|\zeta_{xx}^1\|^2, \\ |I_{33}| \quad & \leq \quad K(K_1, K_5, \epsilon)(\|\zeta^1\|_1^2 + \|\zeta^0\|_1^2) + 3\epsilon\|\zeta_{xx}^1\|^2. \end{aligned}$$

Therefore we have

$$|I_3| \leq K(\epsilon, K_1, K_5)\{k^4 + \|\zeta^0\|_1^2 + \|\zeta^1\|_1^2\} + 8\epsilon\|\zeta_{xx}^1\|^2.$$

By the similar computation as above, we get

$$|I_4| \leq K(\epsilon, K_1, K_2)\{\|\eta^{\frac{1}{2}}\|_{L^\infty}^2\|\eta_{xx}^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|_{W^{1,\infty}}^2\|\eta_x^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|_{L^\infty}^4\} + 6\epsilon\|\zeta_{xx}^1\|^2.$$

And

$$|I_5| \leq K(\rho, \epsilon)\|\eta^{\frac{1}{2}}\|^2 + \epsilon\|\zeta_{xx}^1\|^2.$$

Further we note that

$$(d_t\zeta_x^0, \zeta_x^1) \geq \frac{1}{2k}(\|\zeta_x^1\|^2 - \|\zeta_x^0\|^2).$$

And we obtain the following inequality

$$\begin{aligned}((a(\overline{Z^0})\overline{\zeta_x^0})_x, \zeta_{xx}^1) \quad \geq \quad & \frac{1}{2}\{\tilde{\alpha}\|\zeta_{xx}^1\|^2 - K(\tilde{\alpha}, K_1, K_6)\|\zeta_x^1\|^2 - K(\tilde{\alpha}, K_1, K_6)\|\zeta_x^0\|^2 \\ & -K(\tilde{\alpha}, K_1)\|\zeta_{xx}^0\|^2 - \frac{\tilde{\alpha}}{2}\|\zeta_{xx}^1\|^2\}.\end{aligned}$$

Thus we derive

$$\begin{aligned}&\frac{1}{2k}(\|\zeta_x^1\|^2 - \|\zeta_x^0\|^2) + \frac{1}{4}\tilde{\alpha}\|\zeta_{xx}^1\|^2 \\ &\leq K(\epsilon, K_1, K_2, K_5, K_6, \rho)\{\|d_t\eta^0\|^2 + |\overline{\eta_x^0}(1)|^2 + \|\eta^{\frac{1}{2}}\|_{L^\infty}^2\|\eta_{xx}^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|_{W^{1,\infty}}^2\|\eta_x^{\frac{1}{2}}\|^2 \\ &+\|\eta^{\frac{1}{2}}\|_{L^\infty}^4 + \|\eta^{\frac{1}{2}}\|^2 + k^4\} + K(\epsilon, K_1, K_2, K_5, K_6)\|\zeta^1\|_1^2 + 20\epsilon\|\zeta_{xx}^1\|^2 + \frac{1}{4}\epsilon^2\|\overline{\zeta_{xx}^0}\|^2 \\ &+K(\epsilon)\|d_t u^0 - u_t^{\frac{1}{2}}\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\zeta_x^1\|^2 + \frac{1}{2}\tilde{\alpha}k\|\zeta_{xx}^1\|^2 \\ &\leq K(\epsilon, K_1, K_2, K_4, K_5, K_6, \rho)k\{\|d_t\eta^0\|^2 + |\overline{\eta_x^0}(1)|^2 + \|\eta^{\frac{1}{2}}\|_{L^\infty}^2\|\eta_{xx}^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|^2 \\ &+\|\eta^{\frac{1}{2}}\|_{W^{1,\infty}}^2\|\eta_x^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|_{L^\infty}^4 + k^4\} + kK(\epsilon, K_1, K_2, K_5, K_6)\|\zeta^1\|_1^2 + 40\epsilon k\|\zeta_{xx}^1\|^2 \\ &+\frac{1}{2}\epsilon^2 k\|\overline{\zeta_{xx}^0}\|^2 + K(\epsilon)k\|d_t u^0 - u_t^{\frac{1}{2}}\|^2.\end{aligned}$$

Since

$$\begin{aligned}\|d_t u^0 - u_t^{\frac{1}{2}}\|^2 \quad &= \quad \|\frac{1}{6}k^2 u_{ttt}(\theta_1) - \frac{1}{8}k^2 u_{ttt}(\theta_2)\|^2 \\ &\leq \quad K(K_2)k^4,\end{aligned}$$

we get

$$\|\zeta_x^1\|^2 + \tilde{\beta}k\|\zeta_{xx}^1\|^2 \leq K(\epsilon, K_1, K_2, K_4, K_5, K_6, \rho)k\{h^{2m} + k^4\}$$

for some $\tilde{\beta} > 0, k$ and ϵ sufficiently small.

Now we prove the convergence of ζ^n for $1 \leq n \leq N$.

Theorem 4.2. There exists $K_8 = K_8(K_1, K_2, K_4, K_5, K_6, \rho)$ and h_0 such that for $h \leq h_0, k = O(h), 4 \leq m \leq r + 1 \text{and} \tilde{\beta} > 0$

$$\sup_{1\leq n\leq N} \|\zeta_x^n\|^2 + \tilde{\beta}k\sum_{j=1}^{N}\|\zeta_{xx}^j\|^2 \leq K_8(K_1, K_2, K_4, K_5, K_6, \rho)\{h^{2m} + k^4\}$$

holds.

Proof. By substituting $v = \chi$ into (3.5) and (2.7), we derive the following

(4.6)

$$
\begin{aligned}
&(d_t\tilde{u}^n,\chi_x)+((a(\tilde{u}^{n+\frac12})\tilde{u}_x^{n+\frac12})_x,\chi_{xx})\\
&=-(d_t\eta_x^n,\chi_x)-((\frac{\partial^2 u}{\partial t\partial x})^{n+\frac12}-d_tu_x^n,\chi_x)+a(0)u_x^{n+\frac12}(1)(x\tilde{u}_x^{n+\frac12},\chi_{xx})\\
&+(a_u(u^{n+\frac12})\eta^{n+\frac12}u_x^{n+\frac12})_x,\chi_{xx})-(([a(u^{n+\frac12})-a(\tilde{u}^{n+\frac12})]\tilde{u}_x^{n+\frac12})_x,\chi_{xx})\\
&+\rho({\eta_x}^{n+\frac12},\chi_x).
\end{aligned}
$$

The fifth term on the right hand side of (4.6) can be rewritten as
(4.7)

$$
\begin{aligned}
-(([a(u^{n+\frac12})&-a(\tilde{u}^{n+\frac12})]\tilde{u}_x^{n+\frac12})_x,\chi_{xx})\\
&=((\tilde{a}_u\eta^{n+\frac12}\eta_x^{n+\frac12})_x,\chi_{xx}) - ([a_u(u^{n+\frac12})\eta^{n+\frac12}u_x^{n+\frac12}]_x,\chi_{xx})\\
&\qquad + ([\tilde{a}_{uu}(\eta^2)^{n+\frac12}u_x^{n+\frac12}]_x,\chi_{xx})
\end{aligned}
$$

where

$$
\tilde{a}_u=\int_0^1\frac{\partial a}{\partial u}(u^{n+\frac12}-\xi\eta^{n+\frac12})d\xi,\quad \tilde{a}_{uu}=\int_0^1\frac{\partial^2 a}{\partial u^2}(u^{n+\frac12}-\xi\eta^{n+\frac12})(1-\xi)d\xi.
$$

Substituting (4.7) in (4.6), and subtracting (4.6) from (2.12), we get

$$
\begin{aligned}
(d_t\zeta_x^n,\chi_x)+((a(\widehat{Z^n})\overline{\zeta_x^n})_x,\chi_{xx}) &= -(d_t\eta^n+u_t^{n+\frac12}-d_tu^n,\chi_{xx})\\
&\quad +a(0)[\widehat{Z_x^n}(1)(x\overline{Z_x^n},\chi_{xx})-u_x^{n+\frac12}(1)(x\tilde{u}_x^{n+\frac12},\chi_{xx})]\\
&\quad -((a(\widehat{Z^n})\frac{\tilde{u}_x^n+\tilde{u}_x^{n+1}}{2})_x,\chi_{xx})+((a(\tilde{u}^{n+\frac12})\tilde{u}_x^{n+\frac12})_x,\chi_{xx})\\
&\quad -((\tilde{a}_u\eta^{n+\frac12}\eta_x^{n+\frac12})_x+[\tilde{a}_{uu}(\eta^2)^{n+\frac12}u_x^{n+\frac12}]_x,\chi_{xx})\\
&\quad +\rho(\eta_x^{n+\frac12},\chi_x)\\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

To estimate I_i, $1\le i\le 5$, we take $\chi=\zeta^{n+1}$. Now, we estimate I_1,

$$
|I_1|\le\frac{1}{2\epsilon}(\|d_t\eta^n\|^2+\|d_tu^n-u_t^{n+\frac12}\|^2)+\epsilon\|\zeta_{xx}^{n+1}\|^2.
$$

Since I_2 can be rewritten as

$$
\begin{aligned}
I_2 &= a(0)[\widehat{Z_x^n}(1)(x\overline{\zeta_x^n},\zeta_{xx}^{n+1})+(\widehat{\zeta_x^n}(1)-\widehat{\eta_x^n}(1))(x\overline{\tilde{u}_x^n},\zeta_{xx}^{n+1})+\widehat{u_x^n}(1)(x\overline{\tilde{u}_x^n},\zeta_{xx}^{n+1})\\
&\quad -u_x^{n+\frac12}(1)(x\overline{\tilde{u}_x^n},\zeta_{xx}^{n+1})+u_x^{n+\frac12}(1)(x(\overline{\tilde{u}_x^n}-\tilde{u}_x^{n+\frac12}),\zeta_{xx}^{n+1})],
\end{aligned}
$$

using condition I, (4.1) and Young's inequality, we get

$$
|a(0)[\widehat{Z_x^n}(1)(x\overline{\zeta_x^n},\zeta_{xx}^{n+1})]|\le K(\epsilon,K_1,K_6)(\|\zeta^n\|_1^2+\|\zeta^{n+1}\|_1^2)+\epsilon\|\zeta_{xx}^{n+1}\|^2.
$$

And

$$|(\widehat{\zeta_x^n}(1)-\widehat{\eta_x^n}(1))(x\frac{\tilde{u}_x^{n+1}+\tilde{u}_x^n}{2},\zeta_{xx}^{n+1})| \leq K(\epsilon,K_5)\{\|\widehat{\zeta_x^n}\|^2+|\widehat{\eta_x^n}(1)|^2\}+\frac{1}{4}\epsilon^2\|\widehat{\zeta_{xx}^n}\|^2+\epsilon\|\zeta_{xx}^{n+1}\|^2.$$

From condition II, (3.6) and Young's inequality, we obtain

$$|(\widehat{u_x^n}(1)-u_x^{n+\frac{1}{2}}(1))(x\overline{\tilde{u}^n},\zeta_{xx}^{n+1})| \leq K(\epsilon,K_2,K_5)k^4+\epsilon\|\zeta_{xx}^{n+1}\|^2.$$

Similarly we have

$$|u_x^{n+\frac{1}{2}}(1)(x(\overline{\tilde{u}_x^n}-\tilde{u}_x^{n+\frac{1}{2}}),\zeta_{xx}^{n+1})| \leq K(\epsilon,K_2,K_5)k^4+\epsilon\|\zeta_{xx}^{n+1}\|^2.$$

Therefore,

$$\begin{aligned}|I_2| \quad \leq \quad & K(\epsilon,K_1,K_2,K_5,K_6)\{\|\zeta^n\|_1^2+\|\zeta^{n+1}\|_1^2+\|\widehat{\zeta_x^n}\|^2+|\widehat{\eta_x^n}(1)|^2+k^4\} \\ & +4\epsilon\|\zeta_{xx}^{n+1}\|^2+\frac{1}{4}\epsilon^2\|\widehat{\zeta_{xx}^n}\|^2.\end{aligned}$$

Since I_3 can be rewritten as

$$\begin{aligned}&((a(\tilde{u}^{n+\frac{1}{2}})\tilde{u}_x^{n+\frac{1}{2}}-a(\widehat{Z^n})\overline{\tilde{u}_x^n})_x,\zeta_{xx}^{n+1}) \\ &=((a(\tilde{u}^{n+\frac{1}{2}})\tilde{u}_x^{n+\frac{1}{2}}-a(\widehat{\tilde{u}^n})\tilde{u}_x^{n+\frac{1}{2}}+a(\widehat{\tilde{u}^n})\tilde{u}_x^{n+\frac{1}{2}} \\ &-a(\widehat{\tilde{u}^n})\overline{\tilde{u}_x^n}+a(\widehat{\tilde{u}^n})\overline{\tilde{u}_x^n}-a(\widehat{Z^n})\overline{\tilde{u}_x^n})_x,\zeta_{xx}^{n+1}).\end{aligned}$$

Using condition I, Young's inequality and (3.6), we obtain

$$|I_3| \leq K(\epsilon,K_1,K_5)\{k^4+\|\zeta^n\|_1^2+\|\zeta^{n-1}\|_1^2\}+8\epsilon\|\zeta_{xx}^{n+1}\|^2.$$

Using condition I, Young's inequality and theorem 3.2, we get the following

$$\begin{aligned}|I_4| \quad = \quad & |((\tilde{a}_u\eta^{n+\frac{1}{2}}\eta_x^{n+\frac{1}{2}})_x,\chi_{xx})-([\tilde{a}_{uu}(\eta^2)^{n+\frac{1}{2}}u_x^{n+\frac{1}{2}}]_x,\chi_{xx})| \\ \leq \quad & K(\epsilon,K_1,K_2,K_4)\{\|\eta^{n+\frac{1}{2}}\|_{L^\infty}^2\|\eta_{xx}^{n+\frac{1}{2}}\|^2+\|\eta^{n+\frac{1}{2}}\|_{W^{1,\infty}}^2\|\eta_x^{n+\frac{1}{2}}\|^2+\|\eta^{n+\frac{1}{2}}\|_{L^\infty}^4\} \\ & +6\epsilon\|\zeta_{xx}^{n+1}\|^2.\end{aligned}$$

And

$$|I_5| \leq K(\rho,\epsilon)\|\eta^{n+\frac{1}{2}}\|^2+\epsilon\|\zeta_{xx}^{n+\frac{1}{2}}\|^2.$$

Since

$$(d_t\zeta_x^n,\zeta_x^{n+1}) \geq \frac{1}{2k}(\|\zeta_x^{n+1}\|^2-\|\zeta_x^n\|^2),$$

and

$$((a(\widehat{Z^n})\overline{\zeta_x^n})_x,\zeta_{xx}^{n+1}) \geq \frac{1}{4}\tilde{\alpha}\|\zeta_{xx}^{n+1}\|^2-K(\tilde{\alpha},K_1,K_6)\{\|\zeta_x^{n+1}\|^2+\|\zeta_x^n\|^2+\|\zeta_{xx}^n\|^2\},$$

we finally have

$$\begin{aligned}&\frac{1}{2k}(\|\zeta_x^{n+1}\|^2-\|\zeta_x^n\|^2)+\frac{1}{4}\tilde{\alpha}\|\zeta_{xx}^{n+1}\|^2\\&\leq K(\epsilon,K_1,K_2,K_4,K_5,K_6,\rho)\{\|d_t\eta^n\|^2+|\widehat{\eta_x^n}(1)|^2+\|\eta^{n+\frac{1}{2}}\|_{L^\infty}^2\|\eta_{xx}^{n+\frac{1}{2}}\|^2\\&+\|\eta^{n+\frac{1}{2}}\|_{W^{1,\infty}}^2\|\eta_x^{n+\frac{1}{2}}\|^2+\|\eta^{n+\frac{1}{2}}\|_{L^\infty}^4+\|\eta^{n+\frac{1}{2}}\|^2+k^4\}+20\epsilon\|\zeta_{xx}^{n+1}\|^2+\frac{1}{4}\epsilon^2\|\widehat{\zeta_{xx}^n}\|^2\\&+K(\epsilon)\|d_tu^n-u_t^{n+\frac{1}{2}}\|^2+K(\epsilon,K_1,K_2,K_5,K_6)\{\|\zeta^{n-1}\|_1^2+\|\zeta^n\|_1^2+\|\zeta^{n+1}\|_1^2\}.\end{aligned}$$

Multiplying both sides by $2k$ and using the discrete type Gronwall inequality implies,

$$\begin{aligned}\sup_{2\leq n\leq N}\|\zeta_x^n\|^2+\tilde{\beta}k\sum_{j=2}^{N}\|\zeta_{xx}^j\|^2 \quad&\leq\quad C\{\|\zeta_x^1\|^2+k\|\zeta_{xx}^1\|^2\}\\&\quad+K(\epsilon,K_1,K_2,K_4,K_5,K_6,\rho)(h^{2m}+k^4)\end{aligned}$$

for some $\tilde{\beta}>0$. By applying the result of theorem 4.1 to the inequality above, we obtain

$$\sup_{1\leq n\leq N}\|\zeta_x^n\|^2+\tilde{\beta}k\sum_{j=1}^{N}\|\zeta_{xx}^j\|^2\leq K(\epsilon,K_1,K_2,K_4,K_5,K_6,K_7)\{h^{2m}+k^4\}.$$

Since $e^n=\eta^n-\zeta^n$, by combining the results of theorems 4.1 and 4.2, we get the optimal convergence of e^n in the norms $\|\cdot\|$ and $\|\cdot\|_1$.

Theorem 4.3. There exists $K_9=K_9(K_1,K_2,K_4,K_5,K_6,\rho)$ such that for $4\leq m\leq r+1,\beta>0,k=O(h)$,

$$\begin{aligned}&\sup_{0\leq n\leq N}\|e^n\|_1^2\leq K_9(h^{2(m-1)}+k^4),\\&\sup_{0\leq n\leq N}\|e^n\|_1^2+\beta k\sum_{n=0}^{N-1}\|e^{n+1}\|_2^2\leq K_9(h^{2(m-2)}+k^4),\\&\sup_{0\leq n\leq N}\|e^n\|^2\leq K_9(h^{2m}+k^4)\end{aligned}$$

hold.

Let $e_1^n=s^n-W^n$ and $e_2^n=\tau^n-\tau_h^n$. Now we will estimate the errors $|e_1^n|$ and $|e_2^n|$ in the following theorem.

Theorem 4.4. There exists a constant $K_{10}=K_{10}(K_1,K_2,K_4,K_5,K_6,\rho)$ such that, for $4\leq m\leq r+1$ and $k=O(h)$

$$\sup_{0\leq n\leq N}(|e_1^n|^2+|e_2^n|^2)\leq K_{10}(h^{2m}+k^4).$$

Proof. Subtract (2.15) form (2.5) and multiply the result to get by e_1^{n+1}

(4.8) $\langle d_t e_1^n, e_1^{n+1}\rangle = \langle d_t s^n - (\frac{ds}{dt})^{n+\frac{1}{2}}, e_1^{n+1}\rangle - \langle a(0)u_x^{n+\frac{1}{2}}(1)s^{n+\frac{1}{2}} - a(0)\overline{Z_x^n}(1)\overline{W^n}, e_1^{n+1}\rangle$

where $\langle f,g\rangle = fg$. (4.8) can be splitted into

$$\begin{aligned}\langle d_t e_1^n, e_1^{n+1}\rangle &= \langle d_t s^n - (\frac{ds}{dt})^{n+\frac{1}{2}}, e_1^{n+1}\rangle - \langle a(0)\overline{Z_x^n}(1)\overline{e_1^n}, e_1^{n+1}\rangle \\ &+\langle a(0)(\overline{s^n} - s^{n+\frac{1}{2}})\overline{Z_x^n}(1), e_1^{n+1}\rangle + \langle a(0)s^{n+\frac{1}{2}}(\overline{Z_x^n}(1) - \overline{u_x^n}(1)), e_1^{n+1}\rangle \\ &+\langle a(0)s^{n+\frac{1}{2}}(\overline{u_x^n}(1) - u_x^{n+\frac{1}{2}}(1)), e_1^{n+1}\rangle.\end{aligned}$$

Since

$$\langle d_t e_1^n, e_1^{n+1}\rangle \geq \frac{1}{2k}(|e_1^{n+1}|^2 - |e_1^n|^2),$$

we have

$$\begin{aligned}\frac{1}{2k}(|e_1^{n+1}|^2 - |e_1^n|^2) &\leq |d_t s^n - (\frac{ds}{dt})^{n+\frac{1}{2}}|^2 + K(K_1,K_6)|e_1^{n+1}|^2 + K(K_1,K_6)|e_1^n|^2 \\ &+K(K_1,K_2)|\overline{e_x^n}(1)|^2 + K(K_1,K_2,K_6)k^4, \quad 0 \leq n \leq N-1.\end{aligned}$$

Now, we sum up the terms of the inequality above

$$\begin{aligned}\frac{1}{2k}(|e_1^{n+1}|^2) &\leq \sum_{m=0}^{n}\{|d_t s^m - (\frac{ds}{dt})^{m+\frac{1}{2}}|^2 + K(K_1,K_6)|e_1^{m+1}|^2 + K(K_1,K_6)|e_1^m|^2 \\ &+K(K_1,K_2)(|\overline{\eta_x^n}(1)|^2 + |\overline{\zeta_x^n}(1)|^2) + K(K_1,K_2,K_6)k^4\}, \quad 0 \leq n \leq N-1.\end{aligned}$$

Because of

$$k\sum_{m=0}^{n}|d_t s^m - (\frac{ds}{dt})^{m+\frac{1}{2}}|^2 \leq K(K_2)k^4,$$

using theorem 3.2 and 4.2, we have

$$\begin{aligned}|e_1^{n+1}|^2 &\leq K(K_1,K_2,K_4,K_5,K_6,\rho)k\{h^{2m} + k^4\} \\ &+K(K_1,K_6)k\sum_{m=0}^{n}(|e_1^{m+1}|^2 + |e_1^m|^2), \quad 0 \leq n \leq N-1.\end{aligned}$$

By the application of the discrete Gronwall inequality, we obtain

$$|e_1^{n+1}|^2 \leq K(K_1,K_2,K_4,K_5,K_6,\rho)\{h^{2m} + k^4\}, \quad 0 \leq n \leq N-1.$$

For the estimate for e_2, subtract (2.16) from (2.6) and multiply the result to get by e_2^{n+1},

$$\begin{aligned}\langle d_t e_2^n, e_2^{n+1}\rangle &= \langle d_t \tau^n - (\frac{d\tau}{dt})^{n+\frac{1}{2}}, e_2^{n+1}\rangle - \langle \overline{W^n}, e_2^{n+1}\rangle + \langle (s^{n+\frac{1}{2}})^2, e_2^{n+1}\rangle \\ &\leq |d_t \tau^n - (\frac{d\tau}{dt})^{n+\frac{1}{2}}|^2 + |e_2^{n+1}|^2 - \langle (\overline{W^n})^2 - (\frac{s^n + s^{n+1}}{2})^2, e_2^{n+1}\rangle \\ &+\langle (s^{n+\frac{1}{2}})^2 - (\frac{s^n + s^{n+1}}{2})^2, e_2^{n+1}\rangle.\end{aligned}$$

Since

$$\langle d_t e_2^n, e_2^{n+1} \rangle \geq \frac{1}{2k}(|e_2^{n+1}|^2 - |e_2^n|^2),$$

we have

$$|e_2^{n+1}|^2 \leq kK(K_1, K_2, K_4, K_5, K_6, \rho)(n+1)\{h^{2m} + k^4\} + 6k \sum_{i=1}^{n+1} |e_2^i|^2.$$

Finally we obtain,

$$|e_2^{n+1}|^2 \leq K(K_1, K_2, K_4, K_5, K_6, \rho)\{h^{2m} + k^4\}.$$

Let $U_h^n \equiv Z^n$ and $S_h^n \equiv W^n$. Now we approximate the errors $U^n - U_h^n = U(\tau_h^n) - Z^n$ and $S^n - S_h^n = S(\tau_h^n) - W^n$ in $\|\cdot\|$. By the similar way as in [7], we obtain the following theorem.

Theorem 4.5. Suppose that conditions I and II hold for $\{U, S\}$ and $k = O(h)$. Then for $4 \leq m \leq r+1, \Delta\tau_h = O(h)$ and

$$\sup_n \{\|U^n - U_h^n\|_{L^2(\tilde{\Omega}^n)} + |S^n - S_h^n|\} = K(\nu, \widetilde{K_2}, K_{10})(h^m + (\Delta\tau_h)^2)$$
$$\sup \|U^n - U_h^n\|_{H^1(\tilde{\Omega}^n)} = K(\nu, \widetilde{K_2}, K_9)(h^{m-1} + (\Delta\tau_h)^2)$$

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