J. KSIAM **Vol.3**, No.1, 1-15, 1999

Superconvergent Gradient Recovery For The Parabolic Initial Boundary Value Problem

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on a non-empty closed subset k of a uniformly convex Banach space is proved. The maps f and g satisfy the inequality

$$||fx-gy||^2\phi(||x-fx||\,||y-gy||,\ ||x-gy||,\ ||y-fx||$$

$$||x-fx||\,||x-gy||,\ ||y-fx||\,||y-gy||)$$

$$for\ all\ x,\ y \in k$$

where ϕ is an upper semi-continuous function from R_+^4 to R_+ satisfying(i) $\phi(t,0,\alpha t,0)\beta t$ and $\phi(t,0,0,\alpha t)\beta t)$ where $\beta = 1$ for $\alpha + 2$ and $\beta < 1$ for $\alpha < 2$;(ii) $\phi(t,t,t,t) < t$.

Abstract

Gradient recovery techniques for the second order elliptic boundary value problem are well known. In particular, the Midpoint and the Vertex Recovery Operator have been studied by various authors and under suitable assumptions on the regularity of the unknown solution superconvergence property of these recovered gradients have been proved. In this paper we extend these results to the recovered gradient of the finite element approximation to a model initial-boundary value problem, and go on to prove superconvergence result for this recovered gradient in a discrete (in time) error norm.

1. Introduction. In this paper we study the superconvergence properties of the generic gradient recovery operator ∇^R introduced in Lakhany & Whiteman[3] for the case of model parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} + \mathcal{L}[u] = f \qquad (x,y,t) \in \Omega \times J \tag{1.1a}$$

subject to the initial condition

$$u(x,y,0) = u_0 \qquad (x,y) \in \Omega \tag{1.1b}$$

and the boundary condition

$$u(x,y,t) = 0 \qquad (x,y,t) \in \partial\Omega \times J \tag{1.1c}$$

where J represents the interval $[0, T]$ for some fixed, but otherwise arbitrary, time length T, Ω is a rectangular domain in $\mathbb{R}^2$, $\mathcal{L}$ is a positive definite second order elliptic operator for any $t \in J$, f is sufficiently smooth in $\Omega \times J$. We further assume that Ω is triangulated using a right-angled isosceles triangulation T^h, which essentially means that every element in T^h has as its set of vertices either the set $\{(p, q), (p + h, q), (p, q + h)\}$ or the set $\{(p + h, q + h), (p, q + h), (p + h, q)\}$ for some real numbers p, q such that $(p, q) \in \overline{\Omega}$. We point out that the superconvergent property of the recovered gradients for parabolic problems was first established by Wheeler and Whiteman [9]. In the present paper we shall closely follow their work and moreover support their analysis by means of a numerical experiment.

In dealing with parabolic problems it is convenient to introduce Hilbert spaces of the type $H^p(J; H^r(\Omega))$ for various combinations of non-negative numbers p and r, so that when we say that u is in $H^p(J; H^r(\Omega))$ we imply $u(x,y,t) \in H^r(\Omega) \, \forall t \in J$ and $\|u\|_{H^r(\Omega)}(t) \in H^p(J)$. When no confusion arises regarding the space and time domains over which the above Hilbert spaces are defined we shall, for the sake of brevity, denote such spaces by $H^p(H^r)$ only.

The weak solution $u(x,y,t)$ of the initial-boundary value problem (1.1) satisfies for any $t \in J$ the equality

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + a(t; u, v) = \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega) \tag{1.2}$$

where the bilinear form $a(t; u, v)$ is the one associated with the elliptic operator $\mathcal{L}[u]$ in (1.1a). Its dependence on t comes from the fact that in more general situations $\mathcal{L}[u]$ will include coefficients dependent on the time variable t. The regularity demanded of the weak solution u will be apparent from the analysis carried out in the following sections.

The semi-discrete finite element approximation $u_h(x,y,t)$ for any $t \in J$ satisfies

$$\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle + a(t; u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in S_0^h \tag{1.3}$$

where S_0^h is the finite dimensional subspace of the Hilbert Space $H_0^1(\Omega)$, which we shall take to be the space of continuous piecewise linear functions on the fully structured triangulation T^h of Ω .

The discretization of the time derivative can be done as follows: we first discretize the time interval into M equal parts each of length $k = T/M$ and furthermore define $t_n \equiv nk, n = 1, \cdots M$ and $I_n \equiv (t_{n-1}, t_n)$. We first replace the time derivative in (1.3) by a backward difference approximation after which we apply the finite element method in space over a triangular partition of Ω and obtain the fully discrete approximation $U^n, n = 1, \cdots, M$ through the formulation

$$\left\langle \frac{U^n - U^{n-1}}{k}, V \right\rangle + a\left(t_{n,\theta}; U^{n,\theta}, V\right) = \langle f(\cdot,\cdot,t_{n,\theta}), V \rangle \qquad \forall V \in S_0^h, \qquad n = 1, \cdots M, \tag{1.4}$$

where θ is a parameter such that $0 \leq \theta \leq 1/2$,

$$t_{n,\theta} \equiv \theta t_{n-1} + (1-\theta) t_n \qquad n = 1, \cdots M$$

and for any function ϕ we define

$$\phi^{n,\theta} \equiv \theta \phi^{n-1} + (1-\theta) \phi^n \qquad n = 1, \cdots M.$$

For the complete error analysis of the finite element approximation so obtained we refer to the earlier works of Douglas & Dupont[2], Wheeler[8] and Thomée[7].

In the present paper we shall prove the superconvergence property of the generic recovered gradient function for the general case when θ is allowed to vary between 0 and 1/2 for the fully discrete approximation given in (1.4). The breakdown of this paper is as follows: in §2 we prove a lemma regarding the the approximation property of the elliptic projection of the weak solution u of the initial-boundary value problem (1.1). In §3 we prove the superconvergence property of the recovered gradient $\nabla^R U^n$. Our approach in this section will be similar to that of Wheeler & Whiteman [9]. Finally in §4 we support our study by a numerical experiment.

2. Approximation Properties of Elliptic Projection. In this and the following section we shall consider the initial-boundary value problem (1.1) corresponding to the positive definite elliptic operator

$$\mathcal{L}[u] \equiv -\nabla \cdot A(x,y,t)\nabla u \tag{2.1}$$

with the associated bilinear form

$$a\left(t;u,v\right) = \int_\Omega A(x,y,t)\nabla u \cdot \nabla v dx dy$$

where $A(x,y,t)$ is positive for all values (x,y,t) in the domain $\Omega \times J$. Now for any $t \in J$ the elliptic projection, $\tilde{U} \equiv P_E u$ of u is defined by

$$a\left(t; u-\tilde{U}, \chi\right) = 0 \qquad \forall \chi \in S_0^h. \tag{2.2}$$

We shall make use of the following approximation property of the elliptic projection in our analysis in the sequel.

LEMMA 2.1. *Let u and $\partial u/\partial t$ be in the space $L^2(H^2)$ where u is the weak solution of the initial-boundary value problem (1.1) with $\mathcal{L}[u]$ defined in (2.1). Let $\tilde{U}$ be the elliptic projection of u at any time t given by (2.2), then*

$$\left\| \frac{\partial^\alpha}{\partial t^\alpha}(u-\tilde{U}) \right\|_{L^2(L^2)} \leq Ch^2 \left(\|u\|_{L^2(H^2)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^2)} \right)$$

for $\alpha = 0, 1$.

Proof :. We first recall the following estimate for the elliptic projection (for any $t \in J$):

$$\left\| \nabla(u-\tilde{U}) \right\|_{L^2(\Omega)}(t) \leq Ch\|u\|_{H^2(\Omega)}(t). \tag{2.3}$$

Now for $\alpha = 0$ the proof follows from the Aubin - Nitshe Lemma (c.f. Ciarlet[1]), and as such our main concern is to prove the theorem for the case $\alpha = 1$. Differentiating (2.2) with respect to t we get

$$\int_\Omega \frac{\partial A}{\partial t}\nabla(u-\tilde{U})\cdot\nabla\chi dxdy = -\int_\Omega A\nabla\frac{\partial}{\partial t}(u-\tilde{U})\cdot\nabla\chi dxdy. \tag{2.4}$$

Let $\psi \in S_0^h$ be the elliptic projection of $\partial u/\partial t$ at any time t, then by definition

$$\int_\Omega A\nabla\left[\frac{\partial u}{\partial t}-\psi\right]\cdot\nabla\chi dxdy = 0 \qquad \forall\chi\in S_0^h \tag{2.5}$$

From (2.4) and (2.5), therefore

$$\int_\Omega A\nabla\left[\frac{\partial\tilde{U}}{\partial t}-\psi\right]\cdot\nabla\chi dxdy = \int_\Omega \frac{\partial A}{\partial t}\nabla(u-\tilde{U})\cdot\nabla\chi dxdy. \tag{2.6}$$

Letting $\chi = \partial\tilde{U}/\partial t - \psi$ in (2.6) we obtain using (2.3)

$$\left\|\nabla\left[\frac{\partial\tilde{U}}{\partial t}-\psi\right]\right\|_{L^2(\Omega)}(t) \le Ch\|u\|_{H^2(\Omega)}(t). \tag{2.7}$$

Furthermore, we directly have

$$\left\|\nabla\left[\frac{\partial u}{\partial t}-\psi\right]\right\|_{L^2(\Omega)}(t) \le Ch\left\|\frac{\partial u}{\partial t}\right\|_{H^2(\Omega)}(t). \tag{2.8}$$

From (2.7), (2.8) and the triangle inequality we have

$$\left\|\frac{\partial}{\partial t}\left[\nabla\left(u-\tilde{U}\right)\right]\right\|_{L^2(L^2)} \le Ch\left[\|u\|_{L^2(H^2)}+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^2)}\right]. \tag{2.9}$$

We now estimate the L^2 error $\left\|\frac{\partial}{\partial t}\left[u-\tilde{U}\right]\right\|_{L^2(L^2)}$. In order to do this we consider the following auxiliary boundary value problem for any $t \in J$:

$$\begin{aligned} -\nabla\cdot A(x,y,t)\nabla\phi &= \beta(t) \qquad t\in J, (x,y)\in\Omega \\ \phi &= 0 \qquad (x,y)\in\partial\Omega \end{aligned}$$

where for any $t \in J$, $\beta \in H_0^1(\Omega)$ is such that

$$\int_\Omega A\nabla v\cdot\nabla\beta dxdy = -\int_\Omega \frac{\partial A}{\partial t}\nabla(u-\tilde{U})\cdot\nabla v dxdy \qquad \forall v\in H_0^1(\Omega). \tag{2.10}$$

From (2.10) and (2.4) it follows that

$$\int_\Omega A\nabla\left[\frac{\partial}{\partial t}(u-\tilde{U})-\beta\right]\cdot\chi = 0 \qquad \forall\chi\in S_0^h \tag{2.11}$$

and therefore using Aubin - Nitsche Lemma one can show that

$$\left\|\frac{\partial}{\partial t}(u-\tilde{U})-\beta\right\|_{L^2(\Omega)}(t) \le Ch\left\|\nabla\left[\frac{\partial}{\partial t}(u-\tilde{U})-\beta\right]\right\|_{L^2(\Omega)}(t). \tag{2.12}$$

Furthermore, from (2.10) we see that

$$\|\nabla\beta\|_{L^2(\Omega)}(t) \le C\left\|\nabla(u-\tilde{U})\right\|_{L^2(\Omega)}(t) \tag{2.13}$$

and as such it is clear that, in order to obtain the desired bound, we need an estimate for the term $\|\beta\|_{L^2(\Omega)}(t)$. Using (2.10) we have

$$\begin{aligned}
\|\beta\|^2_{L^2(\Omega)}(t) &= \int_\Omega A\nabla\phi\cdot\nabla\beta dxdy \\
&= -\int_\Omega \frac{\partial A}{\partial t}\nabla(u-\tilde{U})\cdot\nabla\phi dxdy \\
&= -\int_\Omega A\nabla(u-\tilde{U})\cdot\nabla\left[A^{-1}\frac{\partial A}{\partial t}\phi\right]dxdy + \int_\Omega A\nabla(u-\tilde{U})\cdot\phi\nabla\left[A^{-1}\frac{\partial A}{\partial t}\right]dxdy \\
&= -\int_\Omega A\nabla(u-\tilde{U})\cdot\nabla(\alpha\phi-\chi)\,dxdy + \int_\Omega A\nabla(u-\tilde{U})\cdot\phi\nabla\alpha\,dxdy \\
&= T_1 + T_2 \text{ (say)}
\end{aligned} \tag{2.14}$$

where $\alpha = 1/A\ \partial A/\partial t$.

To bound T_1 we make use of (2.3) to obtain

$$|T_1| \le Ch^2\|u\|_{H^2(\Omega)}(t)\|\phi\|_{H^2(\Omega)}(t). \tag{2.15}$$

In order to bound T_2 we first make use of Green's Formula and the fact that ϕ vanishes on the boundary so as to obtain:

$$|T_2| = \left|\int_\Omega (u-\tilde{U})\nabla\cdot A\phi\nabla\alpha dxdy\right| \le Ch^2\|u\|_{H^2(\Omega)}(t)\|\phi\|_{H^2(\Omega)}(t). \tag{2.16}$$

Making use of a stability result in (2.0) and (2.16) and recollecting estimates (2.12), (2.13), (2.14) along with estimates (2.3), (2.9), (2.0) and (2.16) we have the desired proof.

$\blacksquare$

We shall make use of Lemma 2.1 in order to prove Lemma 3.1 in the following section.

3. Superconvergent Error Estimates For The Recovered Gradient. We start this section by proving the following lemma

LEMMA 3.1. *Let u be the weak solution of the initial-boundary value problem (1.1), $\tilde{U}$ denote its elliptic projection in S_0^h at any time $t \in J$ and U^n denote its finite element approximation at time level $t_n = nk$. If θ is the parameter appearing in (1.4) and*

$$|||\cdot|||_{L^2(\Omega)} \equiv \sqrt{\sum_{n=1}^{M} k\|\cdot\|^2_{L^2(\Omega)}}$$

then the following estimate holds

$$\left|\left|\left|\nabla(U^{n,\theta} - \tilde{U}^{n,\theta})\right|\right|\right|_{L^2(\Omega)} \leq Ch^2\left(\|u\|_{L^2(H^2)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^2)}\right)$$
$$+ C\begin{cases} k\left(\left\|\dfrac{\partial^2 u}{\partial t^2}\right\|_{L^2(L^2)} + \left\|\dfrac{\partial u}{\partial t}\right\|_{L^2(H^1)}\right) & 0 \leq \theta < 1/2 \\ k^2\left(\left\|\dfrac{\partial^3 u}{\partial t^3}\right\|_{L^2(L^2)} + \left\|\dfrac{\partial^2 u}{\partial t^2}\right\|_{L^2(H^1)}\right) & \theta = 1/2 \end{cases}$$

provided u is regular enough so that the right hand side is well defined.

Proof:. Letting $v = \chi \in S_0^h$ in (1.2) and replacing the time derivative of u by its backward difference approximation we have, on letting $u^n \equiv u(t_n)$,

$$\left\langle \frac{u^n - u^{n-1}}{k}, \chi \right\rangle + a\left(t_{n,\theta}; u(\cdot,\cdot,t_{n,\theta}), \chi\right) = \left\langle f\left(\cdot,\cdot,t_{n,\theta}\right), \chi\right\rangle + \left\langle R_{n,\theta}, \chi\right\rangle \tag{3.1}$$

where $R_{n,\theta}$ is the Taylor Remainder

$$R_{n,\theta} = \frac{u^n - u^{n-1}}{k} - \frac{\partial u}{\partial t}(\cdot,\cdot,t_{n,\theta}). \tag{3.2}$$

We recall that for any $t \in J$ the elliptic projection, $\tilde{U}$, of u satisfies

$$a\left(t; u - \tilde{U}, \chi\right) = 0 \qquad \forall \chi \in S_0^h. \tag{3.3}$$

Now from (3.1) and (3.3), we have

$$\begin{aligned}\left\langle \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k}, \chi \right\rangle + a\left(t_{n,\theta}; \tilde{U}^{n,\theta}, \chi\right) = &\left\langle \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k} - \frac{u^n - u^{n-1}}{k}, \chi \right\rangle \\ &+ a\left(t_{n,\theta}; \tilde{U}^{n,\theta}, \chi\right) - a\left(t_{n,\theta}; \tilde{U}(\cdot,\cdot,t_{n,\theta}), \chi\right) \\ &+ \left\langle f\left(\cdot,\cdot,t_{n,\theta}\right), \chi\right\rangle + \left\langle R_{n,\theta}, \chi\right\rangle.\end{aligned} \tag{3.4}$$

Subtracting (3.4) from (1.4) we obtain on letting $V = \chi$

$$\begin{aligned}\left\langle \frac{U^n - U^{n-1}}{k} - \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k}, \chi \right\rangle + a\left(t_{n,\theta}; U^{n,\theta} - \tilde{U}^{n,\theta}, \chi\right) = \\ - \left\langle \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k} - \frac{u^n - u^{n-1}}{k}, \chi \right\rangle \\ - a\left(t_{n,\theta}; \tilde{U}^{n,\theta}, \chi\right) + a\left(t_{n,\theta}; \tilde{U}(\cdot,\cdot,t_{n,\theta}), \chi\right) - \langle R_{n,\theta}, \chi \rangle .\end{aligned} \tag{3.5}$$

Setting $\chi = U^{n,\theta} - \tilde{U}^{n,\theta}$ in (3.5), making use of the $H_0^1(\Omega)$-coercivity of the bilinear form $a(t;\cdot,\cdot)$ for all t and noting that with the help of geometric arithmetic mean inequality and the fact that $0 \leq \theta \leq 1/2$, the first term on the left hand side of (3.5) can be written as

$$\begin{aligned}&\left\langle \frac{U^n - U^{n-1}}{k} - \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k}, \theta(U^{n-1} - \tilde{U}^{n-1}) + (1-\theta)(U^n - \tilde{U}^n) \right\rangle \\ &\quad = \frac{1-\theta}{k}\left\|U^n - \tilde{U}^n\right\|^2_{L^2(\Omega)} - \frac{\theta}{k}\left\|U^{n-1} - \tilde{U}^{n-1}\right\|^2_{L^2(\Omega)} \\ &\qquad - \frac{1-2\theta}{k}\left\langle U^n - \tilde{U}^n, U^{n-1} - \tilde{U}^{n-1}\right\rangle \\ &\quad \geq \frac{1}{2k}\left\|U^n - \tilde{U}^n\right\|^2_{L^2(\Omega)} - \frac{1}{2k}\left\|U^{n-1} - \tilde{U}^{n-1}\right\|^2_{L^2(\Omega)}\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2k}\left\{\left\|U^n - \tilde{U}^n\right\|^2_{L^2(\Omega)} - \left\|U^{n-1} - \tilde{U}^{n-1}\right\|^2_{L^2(\Omega)}\right\} + K\left\|\nabla(U^{n,\theta} - \tilde{U}^{n,\theta})\right\|^2_{L^2(\Omega)} \\ &\quad \leq \left|\left\langle \frac{\tilde{U}^n - \tilde{U}^{n-1}}{k} - \frac{u^n - u^{n-1}}{k}, U^{n,\theta} - \tilde{U}^{n,\theta}\right\rangle\right| \\ &\qquad + \left|a\left(t_{n,\theta}; \tilde{U}^{n,\theta}, U^{n,\theta} - \tilde{U}^{n,\theta}\right) - a\left(t_{n,\theta}; \tilde{U}(\cdot,\cdot,t_{n,\theta}), U^{n,\theta} - \tilde{U}^{n,\theta}\right)\right| \\ &\qquad\quad + \left|\left\langle R_{n,\theta}, U^{n,\theta} - \tilde{U}^{n,\theta}\right\rangle\right| \\ &\quad = T_3 + T_4 + T_5 \text{ (say)}.\end{aligned} \tag{3.6}$$

We now estimate terms appearing on the right hand side of (3.6). For instance,

$$\begin{aligned}|T_3| &\leq \left\|\frac{1}{k}\int_{I_n} \frac{\partial}{\partial t}(\tilde{U} - u)dt\right\|_{L^2(\Omega)} \left[\left\|U^n - \tilde{U}^n\right\|_{L^2(\Omega)} + \left\|U^{n-1} - \tilde{U}^{n-1}\right\|_{L^2(\Omega)}\right] \\ &\leq \frac{1}{\sqrt{k}}\left\|\frac{\partial}{\partial t}(\tilde{U} - u)\right\|_{L^2(I_n;L^2(\Omega))} \left[\left\|U^n - \tilde{U}^n\right\|_{L^2(\Omega)} + \left\|U^{n-1} - \tilde{U}^{n-1}\right\|_{L^2(\Omega)}\right]\end{aligned} \tag{3.7}$$

whereupon using Lemma 2.1 we have

$$|T_3| \leq \frac{Ch^2}{\sqrt{k}} \left[\|u\|_{L^2(I_n;H^2(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(I_n;H^2(\Omega))} \right] \left[\left\| U^n - \tilde{U}^n \right\|_{L^2(\Omega)} + \left\| U^{n-1} - \tilde{U}^{n-1} \right\|_{L^2(\Omega)} \right]. \tag{3.8}$$

Next using Taylor Expansion we have

$$\tilde{U}^{n,\theta} = \tilde{U}(\cdot,\cdot,t_{n,\theta}) + \hat{R}_{n,\theta}$$

where

$$\left|\hat{R}_{n,\theta}\right| \leq C \begin{cases} \displaystyle\int_{I_n} \left| \frac{\partial \tilde{U}}{\partial t} \right| dt & 0 \leq \theta < 1/2 \\ \displaystyle k \int_{I_n} \left| \frac{\partial^2 \tilde{U}}{\partial t^2} \right| dt & \theta = 1/2 \end{cases}$$

and therefore using the boundedness of the bilinear form $a\,(t;\cdot,\cdot)$ for any t, we have

$$|T_4| \leq C \left\| \nabla (U^{n,\theta} - \tilde{U}^{n,\theta}) \right\|_{L^2(\Omega)} \begin{cases} k^{1/2} \left\| \nabla \dfrac{\partial \tilde{U}}{\partial t} \right\|_{L^2(I_n;L^2(\Omega))} & 0 \leq \theta < 1/2 \\ k^{3/2} \left\| \nabla \dfrac{\partial^2 \tilde{U}}{\partial t^2} \right\|_{L^2(I_n;L^2(\Omega))} & \theta = 1/2. \end{cases} \tag{3.9}$$

Once again making use of Taylor Expansion, we obtain

$$\begin{aligned} |T_5| &\leq \left\| U^{n,\theta} - \tilde{U}^{n,\theta} \right\|_{L^2(\Omega)} \|R_{n,\theta}\|_{L^2(\Omega)}(t) \\ &\leq C \left\| U^{n,\theta} - \tilde{U}^{n,\theta} \right\|_{L^2(\Omega)} \begin{cases} k^{1/2} \left\| \dfrac{\partial^2 u}{\partial t^2} \right\|_{L^2(I_n;L^2(\Omega))} & 0 \leq \theta < 1/2 \\ k^{3/2} \left\| \dfrac{\partial^3 u}{\partial t^3} \right\|_{L^2(I_n;L^2(\Omega))} & \theta = 1/2. \end{cases} \end{aligned} \tag{3.10}$$

Thus using estimates (3.8), (3.9) and (3.10) we obtain from (3.6)

$$\begin{aligned} &\frac{1}{2k} \left\{ \left\| U^n - \tilde{U}^n \right\|^2_{L^2(\Omega)} - \left\| U^{n-1} - \tilde{U}^{n-1} \right\|^2_{L^2(\Omega)} \right\} + K \left\| \nabla (U^{n,\theta} - \tilde{U}^{n,\theta}) \right\|^2_{L^2(\Omega)} \\ &\leq \left[\frac{C_1 h^2}{\sqrt{k}} \left(\|u\|_{L^2(I_n;H^2(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(I_n;H^2(\Omega))} \right) + \left\{ \begin{array}{ll} k^{1/2} \left\| \dfrac{\partial^2 u}{\partial t^2} \right\|_{L^2(I_n;L^2(\Omega))} & 0 \leq \theta < 1/2 \\ k^{3/2} \left\| \dfrac{\partial^3 u}{\partial t^3} \right\|_{L^2(I_n;L^2(\Omega))} & \theta = 1/2 \end{array} \right\} \right] \end{aligned}$$

$$
\left(\left\|U^n-\tilde{U}^n\right\|_{L^2(\Omega)}+\left\|U^{n-1}-\tilde{U}^{n-1}\right\|_{L^2(\Omega)}\right)
$$

$$
+C_2\left\|\nabla(U^{n,\theta}-\tilde{U}^{n,\theta})\right\|_{L^2(\Omega)}\left\{\begin{array}{ll} k^{1/2}\left\|\nabla\dfrac{\partial\tilde{U}}{\partial t}\right\|_{L^2(I_n;L^2(\Omega))} & 0\le\theta<1/2 \\ k^{3/2}\left\|\nabla\dfrac{\partial^2\tilde{U}}{\partial t^2}\right\|_{L^2(I_n;L^2(\Omega))} & \theta=1/2. \end{array}\right\} \tag{3.11}
$$

Next using the stability of the elliptic projection (c.f. Scott[6]), the arithmetic geometric mean inequality and letting for the sake of brevity

$$
e=U-\tilde{U}
$$

we have from (3.11)

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|e^n\|^2_{L^2(\Omega)}-\left\|e^{n-1}\right\|^2_{L^2(\Omega)}\right\}+K\left\|\nabla e^{n,\theta}\right\|^2_{L^2(\Omega)} \\
&\quad\le\frac{Ch^4}{k}\left(\|u\|_{L^2(I_n;H^2(\Omega))}+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(I_n;H^2(\Omega))}\right)^2 \\
&\qquad+Ck^{2\alpha-1}\left(\left\|\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}\right\|^2_{L^2(I_n;L^2(\Omega))}+\left\|\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right\|^2_{L^2(I_n;H^1(\Omega))}\right) \\
&\qquad\quad+\frac{C_1\epsilon_1}{2}\left(\|e^n\|_{L^2(\Omega)}+\left\|e^{n-1}\right\|_{L^2(\Omega)}\right)^2+\frac{C_2\epsilon_2}{2}\left\|\nabla e^{n,\theta}\right\|^2_{L^2(\Omega)}
\end{aligned} \tag{3.12}
$$

where

$$
\alpha=\begin{cases}1 & 0\le\theta<1/2\\ 2 & \theta=1/2\end{cases}
$$

and ϵ_1 and ϵ_2 are some positive numbers which can be chosen arbitrarily. Multiplying (3.12) throughout by $2k$ and summing over the index n from 1 to M, and using the fact that both ϵ_1 and ϵ_2 can be chosen small enough so that $\exists\gamma>0$ such that

$$
\begin{aligned}
&\gamma\left(\left\|e^M\right\|^2_{L^2(\Omega)}+\sum_{n=1}^{M}k\left\|\nabla e^{n,\theta}\right\|^2_{L^2(\Omega)}\right) \\
&\quad\le Ch^4\left(\|u\|^2_{L^2(H^2)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(H^2)}\right) \\
&\quad+Ck^{2\alpha}\left(\left\|\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}\right\|^2_{L^2(L^2)}+\left\|\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right\|^2_{L^2(H^1)}\right)+C\sum_{n=1}^{M-1}k\|e^n\|^2_{L^2(\Omega)}
\end{aligned} \tag{3.13}
$$

where we have assumed that $U^0 = P_E u_0(x,y)$. Finally using Gronwall's Lemma (c.f. Lees[4]) in (3.13) we obtain the estimate

$$\begin{aligned}\|e^M\|^2_{L^2(\Omega)} &+ \sum_{n=1}^{M} k\|\nabla e^{n,\theta}\|^2_{L^2(\Omega)}\\ &\leq C\left[h^4\left(\|u\|^2_{L(H^2)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L(H^2)}\right) + k^{2\alpha}\left(\left\|\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}\right\|^2_{L^2(L^2)} + \left\|\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right\|^2_{L^2(H^1)}\right)\right]\end{aligned}$$

and hence Lemma 3.1 has been proved.

∎

Lemma 3.1 now allows us to quote the following main result of this paper:

THEOREM 3.2. *Under the conditions of Lemma 3.1 and letting $u \in L^2\left(J; H^3(\Omega)\right)$ we have the following superconvergence estimate:*

$$\left|\left|\left|\nabla u^{n,\theta} - \nabla^R U^{n,\theta}\right|\right|\right|_{L^2(\Omega)} \leq C_1(u)h^2 + C_2(u)k^\alpha$$

where ∇^R is a gradient recovery operator, $\alpha = 1$ if $0 \leq \theta < 1/2$ and $\alpha = 2$ provided $\theta = 1/2$. In the above estimate $C_1(u)$ and $C_2(u)$ are constants dependent on u and its derivatives.

Proof :. The proof follows immediately from Lemma 3.1 and the following superconvergence estimate for the gradient recovery operator ∇^R (c.f Lakhany and Whiteman[3])

$$\left|\left|\left|\nabla u^{n,\theta} - \nabla^R \tilde{U}^{n,\theta})\right|\right|\right|_{L^2(\Omega)} = \sum_{n=1}^{M} k\left\|\nabla u^{n,\theta} - \nabla^R \tilde{U}^{n,\theta})\right\|_{L^2(\Omega)} \leq C(u)h^2.$$

∎

4. A Numerical Experiment. In order to demonstrate the superconvergence property of the Midpoint Recovered Gradient ∇^M (c.f Lakhany and Whiteman[3]) and the effectiveness of the Zienkiewicz - Zhu (or Z^2 or $Z-Z$) Error Estimator (c.f.Zienkiewicz - Zhu[10][11][12], Rodriguez[5]) using this recovered gradient we consider the model initial-boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad (x,y,t) \in [(-1,1)\times(-1,1)]\times[0,1]$$

which has the solution

$$u = \sin^5\left(\frac{\pi t}{2}\right)\exp\left(\frac{-(x^2+y^2)}{0.05}\right) + \frac{(x^2-1)(y^2-1)}{4}$$

dictating appropriate initial and boundary conditions. For this test example the convergence rates for the error at level $t = 1.0$ (for the chosen value of $\theta = 0.0$) are shown in Fig. 4.1 in the L^2-norm, whereas for the same level the convergence in the error $\|\nabla u^n - \nabla U^n\|_{L^2(\Omega)}$ appear in Fig. 4.2. The Zienkiewicz-Zhu Estimator of this error using the Midpoint Recovered Gradient appears in Fig. 4.3. The convergence rates in the norm $|||\nabla(\cdot)|||_{L^2(\Omega)}$ for the gradient of the finite element approximation and the recovered gradient of the same appear in Fig. 4.4 and Fig 4.5 respectively. Finally the Zienkiewicz-Zhu Estimator in the norm $|||\nabla(\cdot)|||_{L^2(\Omega)}$ appears in Fig 4.6. A strong resemblance of the true and computed error justifies the use of Zienkiewicz-Zhu estimator in the case of parabolic problems as well.

Error in L2 Norm at t=1.0

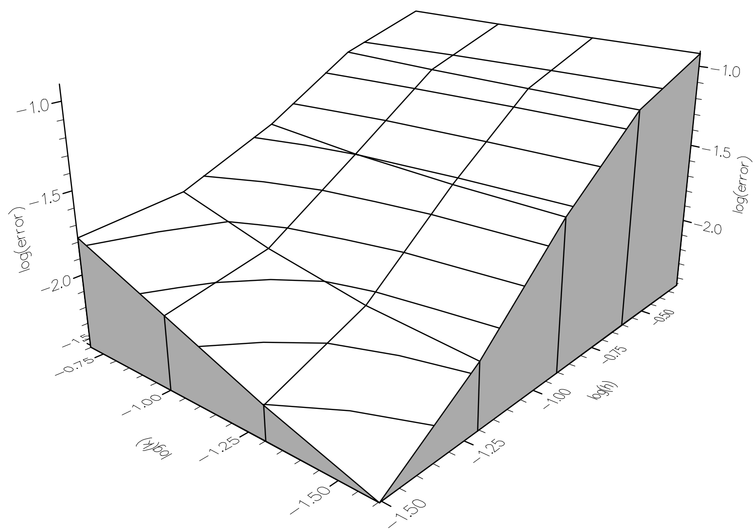

Figure 4.1

Error in H1 Norm at t=1.0

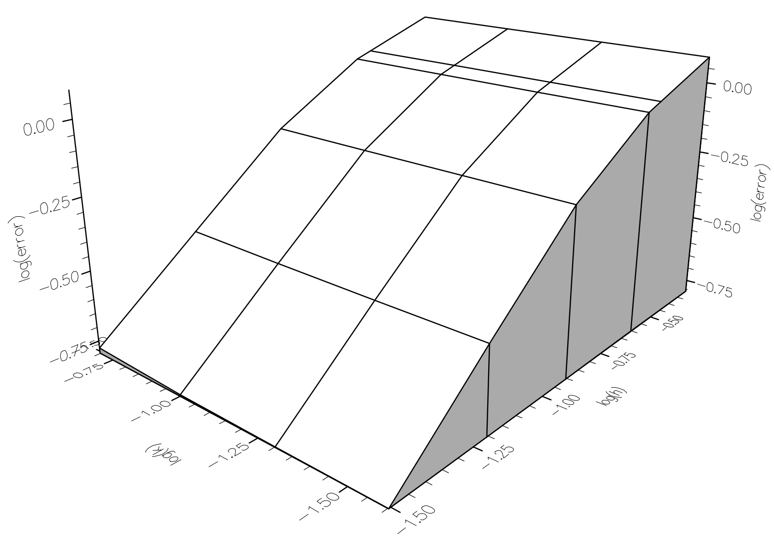

Figure 4.2

Z-Z Estimator at t=1.0

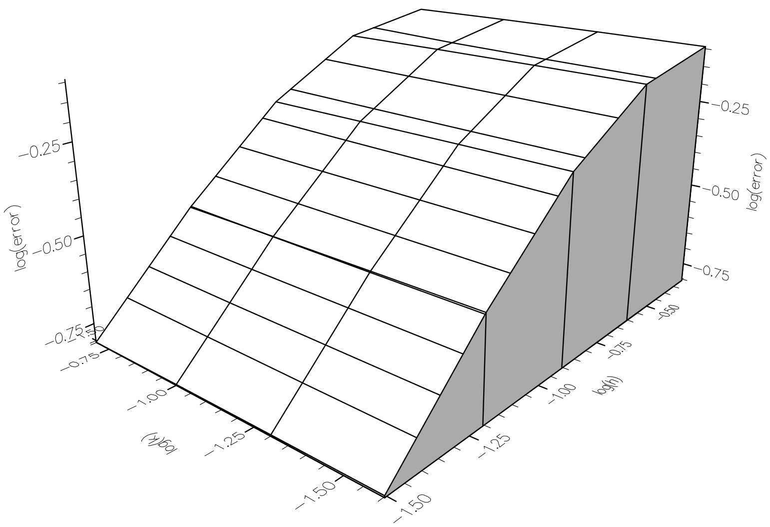

Figure 4.3

Error in Norm ||| |||

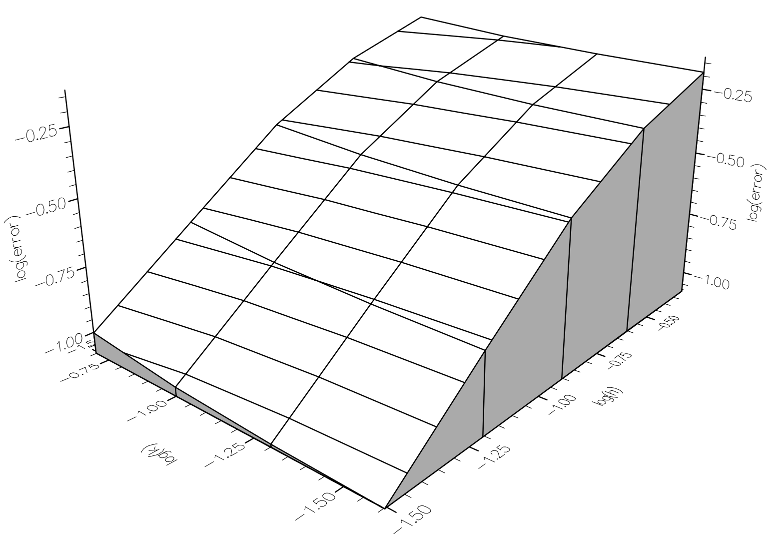

Figure 4.4

Error in Norm ||| ||| for the Recovered Gradient

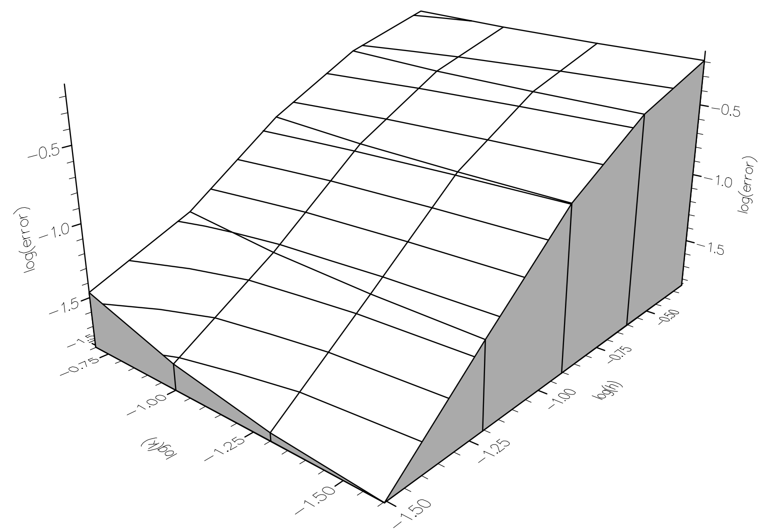

Figure 4.5

Z-Z Estimator in Norm ||| |||

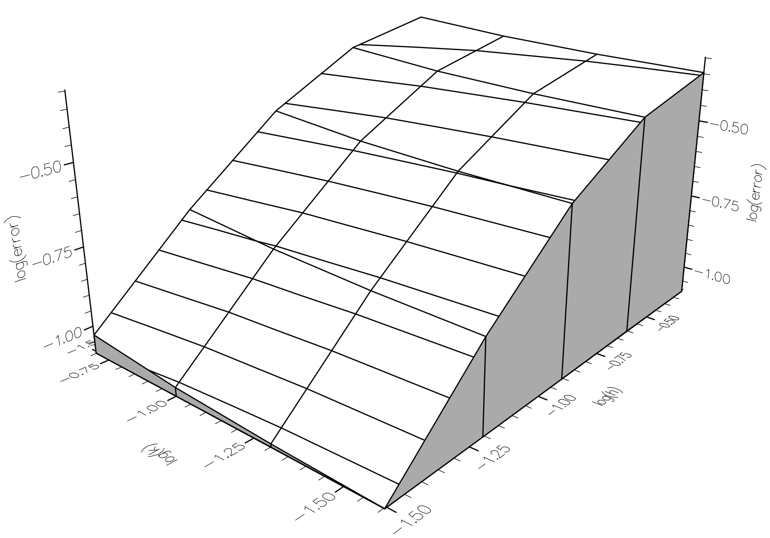

Figure 4.6