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Control Charts for Means and Variances under Multivariate Normal Process

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Abstract

Multivariate quality control charts with combine-accumulate approach and accumulate-combine apprach for monitoring both means and variances under multivariate normal process are investigated. Numerical performances of the charts show that multivariate EWMA chart with accumulate-combine approach can be recommended for all kinds of shift in means and variances.

Key Words and Phrases: ARL, false alarm, accumulate-combine approach

1. Introduction

In many industrial quality control, there exist several correlated quality variables for definig the quality of the product. When the quality variables are correlated, one can obtain better sensitivity by using multivariate control chart than separate control charts for each of the process parameter. The original work on multivariate control chart was introduced by Hotelling(1947). Alt(1984) and Jackson(1985) reviewed much of the studies on multivariate control charts.

During the control process, one wishes to detect any departure from a satisfactory state as quickly as possible and identify which assignable causes are responsible for the deviation. The ability of a control chart is determined by the length of time required for the chart to signal when the process is out-of-control state. Therefore a good control chart should quickly detect shifts in production process parameters while producing few false alarms. The expected time to signal is simply the product of the average run length(ARL) and the length of sampling interval. Therefore, the ARL can be thought of as the expected time to signal.

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Most of the studies on multivariate control charts have been concentrated on monitoring mean vector of multivariate normal process. In this paper, we consider several multivariate control charts to monitor both means and variances of correlated quality variables under multivariate normal process.

2. Constructing Multivariate Sample Statistics

Assume that the process of interest has p $(p \geq 2)$ quality variables represented by the random vector $\underline{X} = (X_1, X_2, \cdots, X_p)'$ and we take a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots$, where $\underline{X}_i = (\underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ is a sample of observations at the sampling time $i (i = 1, 2, \cdots)$ and $\underline{X}_{ij} = (X_{ij1}, \cdots, X_{ijp})'$. It will be also assumed that the successive observation vectors are distributed independent multivariate normal distribution with $N_p(\underline{\mu}, \Sigma)$. Hence, the distribution of $\underline{X}$ is indexed by a set of parameters $\underline{\theta} = (\underline{\mu}, \Sigma)$ where $\underline{\mu}$ is the mean vector and Σ is the dispersion matrix of $\underline{X}$. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the known target values for $\underline{\theta}$.

To control both μ and σ^2 of a single quality variable, Reynolds and Ghosh (1981) proposed an obvious method to use the statistic UV_i and the chart signals if

$$UV_i = \sum_{j=1}^{n} (X_{ij} - \mu_0)^2 / \sigma_0^2 \geq \chi^2_{1-\alpha}(n).$$

Extending the statistic UV_i to multivariate case, we obtain the sample statistic D_i for multivariate chart

$$D_i = \sum_{j=1}^{n} (\underline{X}_{ij} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{X}_{ij} - \underline{\mu}_0). \tag{1}$$

If the process is in-control, D_i has a chi-squared distribution with np degrees of freedom. When the process has changed to $\underline{\mu}$ from the target $\underline{\mu}_0$, D_i has a non-central chi-squared distribution with np degrees of freedom with noncentrality parameter $\tau^2 = n(\underline{\mu} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{\mu} - \underline{\mu}_0)$.

Another control chart for both $\underline{\mu}$ and Σ can be constructed by using both of the likelihood ratio statistic for testing $H_0 : \underline{\mu} = \underline{\mu}_0$ vs $H_1 : \underline{\mu} \neq \underline{\mu}_0$ where Σ_0 is known, and $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$ where $\underline{\mu}_0$ is known.

Likelihood ratio test statistic TM_i at the ith sample $(i = 1, 2, \cdots)$ for testing $H_0 : \underline{\mu} = \underline{\mu}_0$ vs $H_1 : \underline{\mu} \neq \underline{\mu}_0$ can be expressed as

$$TM_i = n(\overline{\underline{X}_i} - \underline{\mu}_0)' \Sigma_0^{-1} (\overline{\underline{X}_i} - \underline{\mu}_0). \tag{2}$$

And, likelihood ratio test statistic TV_i for testing $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$ can be expressed as

$$TV_i = tr(A_i \Sigma_0^{-1}) - n \ln |A_i| + n \ln |\Sigma_0| + np \ln n - np, \tag{3}$$

where $A_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \overline{\underline{X}_i})(\underline{X}_{ij} - \overline{\underline{X}_i})'$.

Thus, the statistics TM_i and TV_i can be used as sample statistics for monitoring $\underline{\mu}$ and Σ, respectively. The statistic TM_i has a chi-squared distribution with p degrees of freedom and the percentage point of TM_i can be obtained from chi-square distribution.

3. Multivariate Shewhart Chart

Shewhart chart, one of the most widely used control charts, has a good ability to detect large changes and is easy to implement the process. For successive random vectors of observation, this chart can be viewed as repeated test of significance.

A multivariate Shewhart chart based on D_i signals whenever

$$D_i \geq \chi^2_{1-\alpha}(np).$$

And a multivariate Shewhart chart based on (TM_i, TV_i) uses separate charts for $\underline{\mu}$ and Σ, and signals if one of the two charts signals whenever

$$TM_i \geq \chi^2_{1-\alpha_{\underline{\mu}}}(p) \quad or \quad TV_i \geq h_{TV},$$

where h_{TV} can be obtained to satisfy a specified in-control ARL by simulation.

4. Multivariate CUSUM Chart

The standard Shewhart chart, although simple to understand and easy to apply, uses only the information in the current sample and is thus relatively inefficient in detecting small or moderate changes. CUSUM charts are often used instead of standard Shewhart chart when detection of small changes in a process parameter is important.

A multivariate CUSUM chart based on the statistic D_i at the ith sample is

$$Y_{D,i} = \max\{Y_{D,i-1}, 0\} + \{D_i - k_D\}, \tag{4}$$

where $Y_{D,0} = \omega_D \cdot I_{(\omega_D \geq 0)}$ and $k_D \geq 0$. This chart signals whenever $Y_{D,i} \geq h_D$.

When the process parameters are on-target or $\underline{\mu}$ shifted, the properties of the multivariate CUSUM chart based on D_i can be evaluated by the Markov chain approach. When the process shifts in Σ, or $\underline{\mu}$ and Σ, the properties of the chart can be evaluated by simulation.

For simultaneous use of (TM_i, TV_i), the multivariate CUSUM control scheme for $\underline{\mu}$ at the ith sample is

$$Y_{TM,i} = \max\{Y_{TM,i-1}, 0\} + (TM_i - k_{TM}) \tag{5}$$

and for Σ at the ith sample is

$$Y_{TV,i} = \max\{Y_{TV,i-1}, 0\} + (TV_i - k_{TV}), \tag{6}$$

where $Y_{TM,0} = \omega_{TM} \cdot I_{(\omega_{TM} \geq 0)}$, $Y_{TV,0} = \omega_{TV} \cdot I_{(\omega_{TV} \geq 0)}$, $k_{TM} \geq 0$ and $k_{TV} \geq 0$. This chart signals whenever $Y_{TM,i} \geq h_{TM}$ or $Y_{TV,i} \geq h_{TV}$.

Since it is difficult to obtain the joint distribution of (5) and (6), the performances of this scheme can be evaluated by simulation when the parameters of the process are on-target or changed.

5. Multivariate EWMA Chart

In EWMA chart, the most recent observations are assigned more weights and the older observations are assigned less weights. The performance of EWMA chart is approximately equivalent to that of CUSUM chart and, in some ways, it is more easier to operate and interpret.

5.1 Combine-Accumulate Approach

Combine-accumulate approach can be constructed by forming multivariate data into a univariate statistic and then accumulates over past sample information. A multivariate EWMA chart for monitoring both $\underline{\mu}$ and Σ based on D_i is

$$Y_{D,i} = (1 - \lambda) Y_{D,i-1} + \lambda D_i, \tag{7}$$

where $Y_{D,0} = \omega \cdot I_{(\omega \geq 0)}$ and $0 < \lambda \leq 1$. This chart signals whenever $Y_{D,i} \geq h_D$.

When the parameters are on-target or $\underline{\mu}$ has changed, the performance of this chart can be evaluated by the Markov chain approach. When the process has changed on Σ, or $\underline{\mu}$ and Σ, the properties of this chart can be evaluated by simulations.

For simultaneous use of (TM_i, TV_i), the multivariate EWMA control scheme for $\underline{\mu}$ at the ith sample is

$$Y_{TM,i} = (1 - \lambda_1) Y_{TM,i-1} + \lambda_1 TM_i \tag{8}$$

and for Σ at the ith sample is

$$Y_{TV,i} = (1 - \lambda_2) Y_{TV,i-1} + \lambda_2 TV_i, \tag{9}$$

where $Y_{TM,0} = \omega_1 \cdot I_{(\omega_1 \geq 0)}$, $Y_{TV,0} = \omega_2 \cdot I_{(\omega_2 \geq 0)}$ and $0 < \lambda_k \leq 1$, $(k = 1, 2)$. This chart signals whenever $Y_{TM,i} \geq h_{TM}$ or $Y_{TV,i} \geq h_{TV}$.

Since it is difficult to obtain the joint distribution of (8) and (9), the performances of this scheme can be evaluated by simulation under the parameters of the process are on-target or changed. The parameters h_D and h_{TM} can be obtained by Markov chain approach, and the parameter h_{TV} can be obtained by simulation.

5.2 Accumulate-Combine Approach

Accumulate-combine approach accumulates past sample information for each process parameter and then combines the separate accumulations into a univariate statistic. Multivariate EWMA chart based on accumulate-combine approach can be constructed by forming a univariate statistic into vectors of EWMA's.

Lowry et al.(1992) proposed a MEWMA chart for $\underline{\mu}$ with accumulate-combine technique. They asserted that MEWMA chart for $\underline{\mu}$ is a more straightforward generalization of the corresponding univariate procedure than the multivariate CUSUM statistics by Crosier(1988) and Pignatiello and Runger(1990). In this section, we propose a procedure which uses two seperate multivariate EWMA charts, based on accumulate-combine approach, for means and for variances. This scheme signals either of these two charts signals.

<u>Chart for Means</u>

MEWMA chart for $\underline{\mu}$, proposed by Lowry et al.(1992), is a multivariate extension of univariate EWMA chart. The vector of EWMA's are defined as

$$\underline{Y_i} = (I - \Lambda)\underline{Y}_{i-1} + \Lambda\underline{\overline{X}}_i, \tag{10}$$

$i = 1, 2, \cdots$, where $\underline{Y}_0 = \underline{\mu}_0$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_P), 0 < \lambda_j \leq 1 (j = 1, \cdots, p)$. Equation (10) can be rewritten by repeated substitution as

$$\underline{Y_i} = \sum_{k=1}^{i} \Lambda(I - \Lambda)^{i-k}\underline{\overline{X}}_i + (I - \Lambda)^i \underline{\mu}_0. \tag{11}$$

This MEWMA chart for $\underline{\mu}$ signals as soon as

$$T_i^2 = (\underline{Y_i} - \underline{\mu_0})'\Sigma_{\underline{Y}_i}^{-1}(\underline{Y_i} - \underline{\mu_0}) > h_1,$$

where h_1 is chosen to achieve a specified in-control ARL by simulation. Under the assumption that $\lambda_1 = \cdots = \lambda_p = \lambda$, the MEWMA vector can be written as

$$\underline{Y}_i = (1 - \lambda)\underline{Y}_{i-1} + \lambda\underline{\overline{X}}_i \tag{12}$$

$i = 1, 2, \cdots$, and the dispersion matrix of $\underline{Y}_i$ is obtained as

$$\Sigma_{\underline{Y}_i} = \frac{\lambda}{2 - \lambda}[1 - (1 - \lambda)^{2i}]\frac{\Sigma}{n}. \tag{13}$$

Lowry et al.(1992) also showed that the distribution of T_i^2 depends on $\underline{\mu}$ and $\sum$ only through the noncentrality parameter τ^2 and small values of λ are more effective in detecting all kinds of shifts in $\underline{\mu}$. Prabhu and Runger(1997) stated that good choices for λ depend on the number of variables in the control scheme and the size of the shift in MEWMA chart for $\underline{\mu}$.

Chart for Variances

The univariate EWMA chart for σ^2 can be constructed by using the statistic

$$Y_k = (1-\lambda)Y_{k-1} + \lambda \sum_{j=1}^{n} \left(\frac{X_{kj} - \mu_0}{\sigma_0} \right)^2, \tag{14}$$

$k = 1, 2, \cdots$ where μ_0, σ_0^2 are known parameters and $0 < \lambda \leq 1$.

Multivariate EWMA chart for variances with accumulate-combine approach can be constructed by suitable modification and forming multivariate extension of the univariate EWMA chart in (14). In multivariate case, we define vectors of EWMA's

$$\underline{Y}_k = \begin{bmatrix} Y_{k1} \\ Y_{k2} \\ \vdots \\ Y_{kp} \end{bmatrix} = \begin{bmatrix} (1-\lambda)^k Y_{10} + \sum_{i=1}^{k} \lambda (1-\lambda)^{k-i} \left[\sum_{j=1}^{n} \left(\frac{X_{ij1} - \mu_{01}}{\sigma_{01}} \right)^2 - n \right] \\ (1-\lambda)^k Y_{20} + \sum_{i=1}^{k} \lambda (1-\lambda)^{k-i} \left[\sum_{j=1}^{n} \left(\frac{X_{ij2} - \mu_{02}}{\sigma_{02}} \right)^2 - n \right] \\ \vdots \\ (1-\lambda)^k Y_{p0} + \sum_{i=1}^{k} \lambda (1-\lambda)^{k-i} \left[\sum_{j=1}^{n} \left(\frac{X_{ijp} - \mu_{0p}}{\sigma_{0p}} \right)^2 - n \right] \end{bmatrix},$$

where $0 < \lambda \leq 1$ and $k = 1, 2, \cdots$.

Then the multivariate EWMA vector can be expressed as

$$\underline{Y}_k = (1-\lambda)\underline{Y}_{k-1} + \lambda \underline{Z}_k, \tag{15}$$

where $\underline{Z}_k = (Z_{k1}, Z_{k2}, \cdots, Z_{kp})'$ and $Z_{ki} = \sum_{j=1}^{n} \left(\frac{X_{ijk} - \mu_{0k}}{\sigma_{0k}} \right)^2 - n$. By repeated substitution in (15) when $\underline{Y}_0 = \underline{0}$, $\underline{Y}_k$ can be written as

$$\underline{Y}_k = \sum_{i=1}^{k} \lambda (1-\lambda)^{k-i} \underline{Z}_i. \tag{16}$$

The dispersion matrix of $\underline{Y}_k$ when a process is in-control and $\underline{Y}_0 = \underline{0}$ is given by

$$\Sigma_{\underline{Y}_k} = \frac{\lambda[1-(1-\lambda)^{2k}]}{2-\lambda} \cdot \Sigma_{\underline{Z}}, \tag{17}$$

where $\Sigma_{\underline{Z}} = 2nR^{(2)}$ and $R^{(2)}$ is used to denote the matrix whose (i,j)th component is the 2^{nd} power of the (i,j)th component of R which is the correlation matrix of

$\underline{X} = (X_1, X_2, \cdots, X_p)$. This multivariate EWMA chart for variance components signals whenever

$$T_k^2 = \underline{Y}_k' \Sigma_{\underline{Y}_k}^{-1} \underline{Y}_k > h_2,$$

where h_2 is choosen to achieve a specified in-control ARL by simulation.

6. Numerical Results and Concluding Remarks

In order to evaluate the properties and compare multivariate Shewhart, CUSUM, and EWMA charts, some kinds of standards for comparison are necessary. For convenience, we let that the sampling interval of unit time is 1 and the ARL of the charts when the process is in-control is fixed to be 200. For simplicity in our computation, we assume that $\underline{\mu}_0 = \underline{0}$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively. And the sample size for each characteristic is five for $p = 2 \sim 4$.

Table 1. ARL values for multivariate charts ($p = 2$)

	C-A [based on D_i]			C-A [based on (TM_i, TV_i)]			A-C
Types of Shift	Shewhart	CUSUM	EWMA	Shewhart	CUSUM	EWMA	EWMA
no shift	200.0	200.0	200.0	200.0	200.1	200.0	200.0
M_1	161.3	136.7	137.0	142.9	115.0	111.4	30.3
M_2	92.5	54.9	62.9	60.7	31.1	34.4	8.9
M_4	20.6	10.1	20.0	9.4	5.2	8.6	2.9
M_6	5.2	4.0	10.5	2.6	2.3	4.1	1.6
M_8	2.0	2.3	6.7	1.3	1.5	2.6	1.2
V_1	82.6	50.6	60.0	134.7	111.7	110.4	46.7
V_2	38.2	20.9	32.3	78.0	56.9	61.0	15.8
V_4	11.9	8.0	17.2	25.3	17.5	24.6	5.2
V_6	5.4	4.7	11.7	10.6	8.3	13.6	2.9
V_8	3.3	3.3	8.9	5.5	5.0	8.9	2.1
(M_1, V_1)	69.2	40.1	50.4	91.8	66.5	67.9	20.4
(M_3, V_3)	9.3	6.6	15.1	9.9	7.2	11.4	3.4
(M_5, V_5)	2.9	3.0	8.2	3.0	2.9	5.0	1.7
(M_7, V_7)	1.6	1.9	5.3	1.7	1.8	3.0	1.3
	$k_D = 11.5$ $\lambda = 0.1$			$(k_{TM}, k_{TV}) = (3.0, 4.5)$ $\lambda = 0.1$			$\lambda = 0.1$

C-A stands for combine-accumulate approach
A-C stands for accumulate-combine approach

The types of shifts in parameters when the process is out-of-control is

1) Shifts in Means : M_i, one mean shifted with $\tau^2 = (i/2)^2$.
2) Shifts in Variances : V_i, one variance is increased to $[1.0 + i/10]^2$.
3) Shifts in Means and Variances : (M_i, V_i) for $i = 1, \cdots, 8$.

After the reference values and the smoothing constants of the proposed charts have been determined, the parameters h's and ARL values were calculated by Markov chain methods or simulation with 10,000 iterations.

Table 2. ARL values for various types of shift ($p = 3$)

	C-A			A-C	
Types of Shift	Shewhart D_i	CUSUM (TM_i, TV_i)	EWMA (TM_i, TV_i)	EWMA	EWMA
no shift	200.0	200.0	199.9	200.0	200.0
M_1	168.9	113.7	123.6	34.9	67.5
M_2	107.0	33.1	42.9	9.9	15.3
M_3	56.6	13.1	18.8	5.0	6.2
M_4	27.8	7.2	10.9	3.1	3.6
M_5	13.7	4.6	7.3	2.2	2.5
M_6	7.1	3.3	5.3	1.7	1.9
M_7	4.1	2.5	4.0	1.4	1.5
M_8	2.6	2.0	3.2	1.2	1.3
V_1	97.4	117.0	126.3	54.8	79.4
V_2	48.9	64.7	75.1	18.1	28.5
V_3	26.4	37.6	48.5	9.1	13.3
V_4	15.6	24.4	33.4	5.7	7.7
V_5	10.0	16.9	24.9	4.1	5.2
V_6	7.0	12.5	19.6	3.2	3.9
V_7	5.1	9.6	15.9	2.6	3.1
V_8	4.0	7.8	13.2	2.2	2.6
(M_1, V_1)	83.8	69.5	80.1	23.6	38.6
(M_2, V_2)	30.9	20.7	28.2	7.2	9.6
(M_3, V_3)	12.5	9.8	14.4	3.7	4.5
(M_4, V_4)	6.0	5.9	9.1	2.4	2.8
(M_5, V_5)	3.5	4.1	6.4	1.8	2.0
(M_6, V_6)	2.4	3.1	4.8	1.5	1.6
(M_7, V_7)	1.8	2.4	3.8	1.3	1.4
(M_8, V_8)	1.5	2.0	3.1	1.2	1.3
	$(k_{TM}, k_{TV}) = (3.5, 9.0)$		$\lambda = 0.1$	$\lambda = 0.1$	$\lambda = 0.3$

Table 3. ARL values for multivariate charts ($p = 4$)

	C-A [based on D_i]			C-A [based on (TM_i, TV_i)]			A-C
Types of Shift	Shewhart	CUSUM	EWMA	Shewhart	CUSUM	EWMA	EWMA
no shift	200.0	200.0	200.0	200.0	200.1	200.0	200.3
M_1	173.4	149.1	153.2	158.3	130.0	131.0	38.6
M_2	116.9	71.2	83.3	84.0	42.4	49.7	10.8
M_4	34.3	15.2	28.9	14.7	7.4	12.9	3.4
M_6	9.1	6.1	15.8	3.7	3.2	6.3	1.8
M_8	3.1	3.4	10.4	1.6	1.9	3.9	1.3
V_1	108.3	67.1	80.0	155.7	130.4	134.5	59.8
V_2	58.4	30.1	45.2	111.3	78.5	86.6	19.8
V_4	19.0	11.8	24.4	50.6	31.7	41.4	6.2
V_6	8.3	6.9	17.1	23.7	16.3	25.4	3.3
V_8	4.6	4.7	13.1	12.4	10.2	17.6	2.3
(M_1, V_1)	94.4	54.6	69.1	116.5	85.6	90.6	26.3
(M_3, V_3)	15.3	9.9	21.8	15.4	10.4	17.0	4.0
(M_5, V_5)	4.2	4.4	12.4	4.2	4.0	7.6	1.9
(M_7, V_7)	2.0	2.7	8.2	2.1	2.4	4.5	1.4
	$k_D = 21.5$ $\lambda = 0.1$			$(k_{TM}, k_{TV}) = (5.0, 16.5)$ $\lambda = 0.1$			$\lambda = 0.1$

The numerical results of the proposed charts were stated in Tables 1-3. In combine-accumulate approach, the performance of multivariate EWMA schemes is approximately equivalent to that of CUSUM schemes when small or moderate changes in the process have occurred. Numerical results for various reference values show that large reference values are efficient for large shifts and vice versa in multivariate CUSUM charts. In multivariate EWMA charts, small smoothing constants are efficient for small or moderate shifts and large smoothing constants are efficient for large shifts. Our numerical results also show that multivariate EWMA chart based on accumulate-combine approach can be recommended for monitoring both means and variances.

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