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Bayesian Tests for Independence and Symmetry in Freund's Bivariate Exponential Model

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Abstract

In this paper, we consider the Bayesian hypotheses testing for independence and symmetry in Freund's bivariate exponential model. In Bayesian testing problem, we use the noninformative priors for parameters which are improper and are defined only up to arbitrary constants. And we use the recently proposed hypotheses testing criterion called the intrinsic Bayes factor. Also we derive the arithmetic and median intrinsic Bayes factors and use these results to analyze some data sets.

Key Words and Phrases: Intrinsic Bayes Factor, Bayesian Testing, Freund's Model.

1. Introduction

Let's consider a life testing experiment in which multiple two-component shared parallel systems are put on test. In many cases of life testing and reliability analysis, components are assumed to have independent life distributions. However, in many life testing situations it is more realistic to assume some form of positive dependence among components. This positive dependece among component life lengths arises from common environmental stresses and shocks, from components depending on common sources of power, and so on. As an example, we consider the paired organs like kidneys, eyes, ears or any other paired organs in an individual as two component system. In these cases, each paired organ is correlated each other. Freund(1961) formulated a bivariate extension of the exponential model as a model for a system where the failure times of the two components may depend on each other.

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Kunchur and Munoli(1994) obtained minimum variance unbiased estimator for the system reliability. Weier(1981), Hanagal and Kale(1992), Hanagal(1996) et al. studied Freund's model with complete data set.

In Bayesian testing problem, the Bayes factor under proper priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. But noninformative priors such as Jeffrey's(1961) or reference priors (Berger and Bernardo(1989,1992)) are typically improper so that the priors are defined only up to arbitrary constants which affects the values of Bayes factors. So, Geisser and Eddy(1979), Spiegelhalter and Smith(1982), San Martini and Spezzaferri(1984) and O'Hagan(1995) have made efforts to compensate for that arbitrariness.

Berger and Pericchi(1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factor(IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. This approach has shown to be quite useful (Berger and Pericchi(1996a), Varshavsky(1996) and Lingham and Sivaganesan(1997)).

In this paper, we consider a Bayesian approach to test independence and symmetry in Freund's bivariate exponential model. Here we use noninformative priors as improper priors. Also we derive intrinsic Bayes factors to solve our problem and give some numerical results to illustrate our results.

2. Preliminaries

Let the random variables (X, Y) follow Freund's bivariate exponential model with parameters $\theta = (\alpha, \alpha', \beta, \beta')$. Then the joint probability density function is given as

$$f(x, y : \theta) = \begin{cases} \alpha\beta' \exp\left[-\beta' y - (\alpha + \beta - \beta')x\right], & y > x > 0, \\ \alpha'\beta \exp\left[-\alpha' x - (\alpha + \beta - \alpha')y\right], & x > y > 0. \end{cases} \tag{2.1}$$

Now, we introduce the intrinsic Bayes factor in the general hypotheses testing. As a matter of convenience, we introduce the following notations.

$\mathbf{X} = (X_1, \cdots, X_n)$: observation with density $f(\mathbf{x}|\theta)$, where $\theta \in \Theta$ is a finite dimensional parameter and Θ is parameter space.
Θ_i : parameter space under ith hypothesis H_i , $i = 1, 2, \cdots, q$.
$f(x|\theta_i)$: the density under H_i , $i = 1, 2, \cdots, q$.
$\pi_i(\theta_i)$: the prior distribution under H_i , $i = 1, 2, \cdots, q$.
$m_i(\mathbf{x})$: the marginal density of $\mathbf{X}$ under H_i when use $\pi_i(\theta_i), i = 1, 2, \cdots, q$.
p_i : the prior probability of H_i being true, $i = 1, 2, \cdots, q$.

$\pi_i^N(\theta_i)$: the improper prior distribution under H_i , $i = 1, 2, \cdots, q$.
$m_i^N(\mathbf{x})$: the marginal density of $\mathbf{X}$ under H_i when use $\pi_i^N(\theta_i), i = 1, 2, \cdots, q$.

Then $\pi_i^N(\theta_i)$ is usually written as $\pi_i^N(\theta_i) \propto h_i(\theta_i)$, where h_i is a function whose integral over the Θ_i-space diverges. Formally, we can write $\pi_i^N(\theta_i) = c_i \, h_i(\theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant.
The posterior probability that H_i is true is given as

$$P(H_i|\mathbf{x}) = (\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji})^{-1} \tag{2.2}$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})} = \frac{\int_{\Theta_j} f(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int_{\Theta_i} f(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i}. \tag{2.3}$$

The posterior probabilities in (2.2) are then used to select the most plausible hypothesis. If one were to use some noninformative priors, then (2.3) becomes

$$B_{ji}^N = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})} = \frac{\int_{\Theta_j} f(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j}{\int_{\Theta_i} f(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}. \tag{2.4}$$

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminancy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, \ i = 1, \cdots, q. \tag{2.5}$$

In view (2.5), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\mathbf{x}(-l)$, using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))\,d\theta_j}{\int_{\Theta_i} f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))\,d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.6}$$

where B_{ji}^N is given by (2.4) and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}. \tag{2.7}$$

In (2.6), any arbitrary ratio, c_j/c_i say, that multiples B_{ji}^N would be cancelled by the ratio c_i/c_j forming the multiplicand in $B_{ij}^N(\mathbf{x}(l))$. Also, while the expression (2.7) renders $B_{ji}(l)$ in terms of the simpler marginal densties of $\mathbf{x}(l)$.

As training samples, Arithmetic and Median Intrinsic Bayes Factor play a fundamental role in our testing $H_i, i = 1, \cdots, q$, we introduce the following definitions.

Definition 1.(Berger and Pericchi(1996b)) A training sample $\mathbf{x}(l)$, will called *proper* if (2.5) holds and *minimal* if it is proper and none of its subsets is proper.

Definition 2.(Berger and Pericchi(1996b)) The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^{N} \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^{N}(\mathbf{x}(l)) \tag{2.8}$$

where L is the number of all possible minimal training samples.

Definition 3.(Berger and Pericchi(1998)) The Median Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot ME[B_{ij}^{N}(\mathbf{x}(l))] \tag{2.9}$$

where $ME[B_{ij}^{N}(\mathbf{x}(l))]$ indicates the median, here to be taken over all the training sample Bayes factors.

We can also calculate the posterior probability of H_i using (2.5), where B_{ji} are replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.8) and (2.9).

3. Bayesian Hypotheses Tests

In Freund's bivariate exponential model, we want to test the hypotheses of symmetry and independence test. That is, $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. H_2 : *not* H_1 and $H_3 : \alpha = \beta$ and $\alpha' = \beta'$ v.s. H_4 : *not* H_3. Consider samples of sizes n from Freund's model with parameters $\theta = (\alpha, \alpha', \beta, \beta')$.

3.1 Symmetry Test

The goal here is to determine the set of all possible minimal training sample for the data $(\mathbf{x}, \mathbf{y})$ to test $H_1 : \alpha = \beta, \alpha' = \beta'$ v.s. H_2 : *not* H_1. Here, let $\theta_1 = (\alpha, \alpha')$ and $\theta_2 = (\alpha, \alpha', \beta, \beta')$. The noninformative priors for $H_1 : \alpha = \beta, \alpha' = \beta'$ v.s. H_2 : *not* H_1 are respectively given by

$$\pi_1^N(\theta_1) = \frac{1}{\alpha\alpha'} \tag{3.1}$$

and

$$\pi_2^N(\theta_2) = \frac{1}{\alpha\alpha'\beta\beta'}. \tag{3.2}$$

To derive the marginals with respect to the noninformative priors given by (3.1) and (3.2), we first observe that the joint pdf of $(\mathbf{x}, \mathbf{y})$ is given by

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) &= \prod_{i=1}^{n}[f(x_i,y_i)]^{(R_i+R_i^*)} \\ &= \alpha^{n_1}\cdot\beta^{n_2}\cdot\alpha_2'^{n}\cdot\beta_1'^{n}\cdot\exp\left[-\alpha(\sum_{i\in S_1}x_i+\sum_{i\in S_2}y_i)\right]\cdot\exp\left[-\beta(\sum_{i\in S_1}x_i+\sum_{i\in S_2}y_i)\right] \\ &\quad\cdot\exp\left[-\alpha'(\sum_{i\in S_2}x_i-\sum_{i\in S_2}y_i)\right]\cdot\exp\left[-\beta'(\sum_{i\in S_1}(y_i-x_i))\right] \end{aligned} \tag{3.3}$$

where $R_i = I(X_i < Y_i)$, $R_i^* = 1 - R_i$, $i = 1, 2, \cdots, n$ $n_1 = \sum_{i=1}^{n} R_i$, $n_2 = \sum_{i=1}^{n} R_i^*$, $S_1 = \{i|R_i = 1,\ i = 1, 2, \cdots, n\}$ $S_2 = \{i|R_i^* = 1,\ i = 1, 2, \cdots, n\}$.

Moreover, the joint pdf of any four paired observations, (x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m) , $1 \leq j < k < l < m \leq n$, is given by

$$\begin{aligned} \prod_{i\in\{j,k,l,m\}} f(x_i,y_i) &= \prod_{i\in\{j,k,l,m\}}\{[f(x_i,y_i)]^{R_i}[f(x_i,y_i)]^{R_i^*}\} \\ &= \alpha^{n'_1}\cdot\beta^{n'_2}\cdot\alpha'^{n'_2}\beta'^{n'_1}\exp\left[-\alpha(\sum_{i\in S'_i}x_i+\sum_{i\in S'_2}y_i)\right] \\ &\quad\cdot\exp\left[-\beta(\sum_{i\in S'_i}x_i+\sum_{i\in S'_2}y_i\right]\cdot\exp\left[-\alpha(\sum_{i\in S'_i}x_i-\sum_{i\in S'_2}y_i)\right] \\ &\quad\cdot\exp\left[-\beta'(\sum_{i\in S'_1}(y_i-x_i)\right]. \end{aligned} \tag{3.4}$$

Here, $n'_1 = \sum_{i\in\{j,k,l,m\}} R_i$, $n'_2 = \sum_{i\in\{j,k,l,m\}} R_i^*$, $S'_1 = \{i|R_i = 1,\ i \in \{j, k, l, m\}\}$, and $S'_2 = \{i|R_i^* = 1,\ i \in \{j, k, l, m\}\}$.

In the following lemma, we give the marginal densities for any three paired observations.

Lemma 1. For the minimal training sample case, we have the marginal density $m_i^N((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$ under H_i, $i = 1, 2$ as follows.

$$m_1^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m)) \tag{3.5}$$
$$= [\Gamma(n'_1+n'_2)]^2 \cdot \left(\frac{1}{\sum_{i\in S'_1} x_i + \sum_{i\in S'_2} x_i}\right)^{n'_1+n'_2} \cdot \left(\frac{1}{\sum_{i\in S'_1} y_i + \sum_{i\in S'_2} y_i}\right)^{n'_1+n'_2}$$

and

$$m_2^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m)$$
$$= [\Gamma(n'_2)\cdot\Gamma(n'_1)]^2 \cdot \left(\frac{1}{\sum_{i\in S'_1} x_i + \sum_{i\in S'_2} y_i}\right)^{n'_1} \cdot \left(\frac{1}{\sum_{i\in S'_1} x_i + \sum_{i\in S'_2} y_i}\right)^{n'_2}$$
$$\cdot \left(\frac{1}{\sum_{i\in S'_2} x_i - \sum_{i\in S'_2} y_i}\right)^{n'_2} \cdot \left(\frac{1}{\sum_{i\in S'_1}(y_i - x_i)}\right)^{n'_1}. \tag{3.6}$$

Since the marginal densities $m_1^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ and $m_2^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ are finite for all $1 \le j < k < l < m \le n$ under each hypothesis, we conclude that any training sample of size three is a minimal training sample.

The marginal densities corresponding to the full data $(\mathbf{X},\mathbf{Y})$ for test $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. $H_2 :$ *not* H_1 can also be expressed in the following lemma.

Lemma 2. For the full data, we have the marginal density $m_i^N(\mathbf{x},\mathbf{y})$ under $H_i, i = 1,2$ as follows.

$$m_1^N(\mathbf{x},\mathbf{y}) = [\Gamma(n_1+n_2)]^2 \cdot \left(\frac{1}{\sum_{i\in S_1} x_i + \sum_{i\in S_2} x_i}\right)^{n_1+n_2} \cdot \left(\frac{1}{\sum_{i\in S_1} y_i + \sum_{i\in S_2} y_i}\right)^{n_1+n_2} \tag{3.7}$$

and

$$m_2^N(\mathbf{x},\mathbf{y}) = [\Gamma(n_1)\cdot\Gamma(n_2)]^2 \cdot \left(\frac{1}{\sum_{i\in S_1} x_i + \sum_{i\in S_2} y_i}\right)^{n_1} \cdot \left(\frac{1}{\sum_{i\in S_1} x_i + \sum_{i\in S_2} y_i}\right)^{n_2}$$
$$\cdot \left(\frac{1}{\sum_{i\in S_2} x_i - \sum_{i\in S_2} y_i}\right)^{n_2} \cdot \left(\frac{1}{\sum_{i\in S_1} y_i - \sum_{i\in S_1} x_i}\right)^{n_1}. \tag{3.8}$$

To test $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. $H_2 :$ *not* H_1, we get the following theorem from Lemmas 1 and 2.

Theorem 1. (i) The Bayes factor using the full data is given by

$$B_{21}^{N}=\frac{m_2^N((\mathbf{x},\mathbf{y}))}{m_1^N((\mathbf{x},\mathbf{y}))}. \tag{3.9}$$

(ii) The Bayes factor using the $(\mathbf{x},\mathbf{y})(l)=((x_j,y_j),\ (x_k,y_k),\ (x_l,y_l),\ (x_m,y_m))$ is given by

$$B_{12}^{N}((\mathbf{x},\mathbf{y})(l))=\frac{m_1^N((\mathbf{x},\mathbf{y})(l))}{m_2^N((\mathbf{x},\mathbf{y})(l))}. \tag{3.10}$$

From the Theorem 1, the arithmetic intrinsic Bayes factor B_{21}^{AI} to test $H_1\ :\ \alpha=\beta,\ \alpha'=\beta'$ v.s. $H_2\ :\ not\ H_1$ is given by

$$B_{21}^{AI}=B_{21}^{N}\cdot\frac{1}{\binom{n}{4}}\sum_{l}B_{12}^{N}((\mathbf{x},\mathbf{y})(l)). \tag{3.11}$$

Next we use the another intrinsic Bayes factor called median intrinsic Bayes factor (Berger and Pericchi(1998)). They showed that the median intrinsic Bayes factor seems to be a simple and very generally applicable intrinsic Bayes factor, which works well for nested or non-nested models, and even for small or moderate sample sizes.

From the Definition 3, Lemma 1, Lemma 2 and Theorem 1, we derive the median intrinsic Bayes factor to test $H_1\ :\ \alpha=\beta,\ \alpha'=\beta'$ v.s. $H_2\ :\ not\ H_1$ as follow:

$$B_{21}^{MI}=B_{21}^{N}\cdot ME[B_{12}^{N}((\mathbf{x},\mathbf{y})(l))] \tag{3.12}$$

3.2 Independence Test

The goal here is to determine the set of all possible minimal training sample for the data $(\mathbf{x},\mathbf{y})$ to test $H_3\ :\ \alpha=\alpha',\ \beta=\beta'$ v.s. $H_4\ :\ not\ H_3$.

To test $H_3\ :\ \alpha=\alpha',\ \beta=\beta'$ v.s. $H_4\ :\ not\ H_3$, we must to determine the set of all possible minimal training samples for the data $(\mathbf{x},\mathbf{y})$. Here, let $\theta_3=(\alpha,\beta)$ and $\theta_4=(\alpha,\alpha',\beta,\beta')$. The noninformative priors for $H_3\ :\ \alpha=\alpha',\ \beta=\beta'$ v.s. $H_4\ :\ not\ H_3$ are respectively given by

$$\pi_3^N(\theta_3)=\frac{1}{\alpha\beta} \tag{3.13}$$

and

$$\pi_4^N(\theta_4)=\frac{1}{\alpha\alpha'\beta\beta'}. \tag{3.14}$$

In the following lemma, we now derive the marginals with respect to the noninformative priors given by (3.13) and (3.14).

Lemma 3. We have the marginal density $m_i^N((x_j,y_j),(x_k,y_k),(x_l,y_l),\ (x_m,y_m))$, under $H_i,\ i=3,4$ as follows.

$$\begin{aligned} & m_3^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m)) \\ & = [\Gamma(n'_1+n'_2)]^2 \left(\frac{1}{\sum_{i\in S'_1} x_i + \sum_{i\in S'_2} x_i}\right)^{n'_1+n'_2} \cdot \left(\frac{1}{\sum_{i\in S'_1} y_i + \sum_{i\in S'_2} y_i}\right)^{n'_1+n'_2} \end{aligned} \tag{3.15}$$

and $m_4^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ is the same as $m_2^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ of Lemma 1.
Also, we conclude that any training sample of size three is an MTS.

Nextly the marginal densities corresponding to the full data $(\mathbf{X},\mathbf{Y})$ for test $H_3\ :\ \alpha=\alpha',\ \beta=\beta'$ v.s. $H_4\ :\ not\ H_3$ can also be expressed in the following lemma.

Lemma 4. For the full data, we have the marginal density $m_i^N(\mathbf{x},\mathbf{y}),\ i=3,4$ under $H_i,\ i=3,4$ as follow.

$$\begin{aligned} m_3^N((\mathbf{x},\mathbf{y})) &= [\Gamma(n_1+n_2)]^2 \cdot \left(\frac{1}{\sum_{i\in S_1} x_i + \sum_{i\in S_2} x_i}\right)^{n_1+n_2} \\ &\quad \cdot \left(\frac{1}{\sum_{i\in S_1} y_i + \sum_{i\in S_2} y_i}\right)^{n_1+n_2}. \end{aligned} \tag{3.16}$$

and $m_4^N((\mathbf{x},\mathbf{y}))$ is the same as $m_2^N((\mathbf{x},\mathbf{y}))$ of Lemma 2.

Nextly we get the following theorem from Lemmas 3 and 4.

Theorem 2. (i) The Bayes factor using the full data is given by

$$B_{43}^N = \frac{m_4^N((\mathbf{x},\mathbf{y}))}{m_3^N((\mathbf{x},\mathbf{y}))}. \tag{3.17}$$

(ii) The Bayes factor using the $(\mathbf{x},\mathbf{y})(l) = ((x_j,y_j),\ (x_k,y_k),\ (x_l,y_l),\ (x_m,y_m))$ is given by

$$B_{34}^N((\mathbf{x},\mathbf{y})(l)) = \frac{m_3^N((\mathbf{x},\mathbf{y})(1))}{m_4^N((\mathbf{x},\mathbf{y})(l))}. \tag{3.18}$$

From the Theorem 2, the arithmetic and median intrinsic Bayes factor B_{43}^{AI} to test $H_3\ :\ \alpha=\alpha',\ \beta=\beta'$ v.s. $H_4\ :\ not\ H_3$ is given by

$$B_{43}^{AI} = B_{43}^N \cdot \frac{1}{\binom{n}{4}} \sum_l B_{34}^N((\mathbf{x},\mathbf{y})(l)) \tag{3.19}$$

and

$$B_{43}^{MI} = B_{43}^N \cdot ME[B_{34}^N((\mathbf{x},\mathbf{y})(l))]. \tag{3.20}$$

4. Simulation Study

In this section, we present two examples to illustrate for our test (i) $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. H_2 : *not* H_1 and (ii) $H_3 : \alpha = \alpha', \beta = \beta'$ v.s. H_4 : *not* H_3. When model uncertainty is assesed, we can take the prior probability of H_i being true, $p_i = 0.5,\ i = 1, 2$ and $i = 3, 4$, respectively.

Example 1 : The data given below are simulated data of size 10 from Freund's bivariate exponential model with parameters $(\alpha, \alpha', \beta, \beta') = (0.1, 0.11, 0.5, 0.51)$.

i	1	2	3	4	5	6	7	8	9	10
x_i	.0397	.0954	.1595	.0095	.0044	.1448	.0552	.0852	.0386	.0392
y_i	.0658	.0890	.1195	.4834	.3926	.0050	.8101	.3798	.0719	.4821

For above data, Table 1 indicates Bayes factors and $P(H_1|\mathbf{x}, \mathbf{y})$ for testing $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. H_2 : *not* H_1.

Table 1 : Bayes factors and $P(H_1|\mathbf{x}, \mathbf{y})$ for testing $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. $H_2 :$ *not* H_1.

B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_1|\mathbf{x}, \mathbf{y})$	$P^{MI}(H_1|\mathbf{x}, \mathbf{y})$
14.8977	24.9908	0.0629	0.0384

Also, for above data Table 2 indicates Bayes factors and $P(H_3|\mathbf{x}, \mathbf{y})$ for testing $H_3 : \alpha = \alpha',\ \beta = \beta'$ v.s. H_4 : *not* H_3.

Table 2 : Bayes factors and $P(H_3|\mathbf{x}, \mathbf{y})$ for testing $H_3 : \alpha = \alpha',\ \beta = \beta'$ v.s. $H_4 :$ *not* H_3.

B_{431}^{AI}	B_{43}^{MI}	$P^{AI}(H_3|\mathbf{x}, \mathbf{y})$	$P^{MI}(H_3|\mathbf{x}, \mathbf{y})$
.5257	.9555	.6554	.5113

From table 1, since $B_{21}^{AI} = 14.8977,\ P^{AI}(H_1|\mathbf{x}, \mathbf{y}) = 0.0629$ and $B_{21}^{MI} = 24.9908$, $P^{MI}(H_1|\mathbf{x}, \mathbf{y}) = 0.0384$, there is strong evidence for H_2 and H_1 in terms of the posterior probability, respectively. That is, there is strong evidence for non-symmetry for above bivariate data in terms of the posterior probability $P^{AI}(H_2|\mathbf{x}, \mathbf{y}) = 0.9371$ and $P^{MI}(H_2|\mathbf{x}, \mathbf{y}) = 0.9616$.

From table 2, since $B_{43}^{AI} = 0.5257,\ P^{AI}(H_3|\mathbf{x}, \mathbf{y}) = 0.6554$ and $B_{43}^{MI} = 0.9555$, $P^{MI}(H_3|\ \mathbf{x}, \mathbf{y}) = 0.5113$, there is no strong evidence for H_4 and H_3 in terms of the posterior probability, respectively. That is, there is no strong evidence for independence for above bivariate data in terms of the posterior probability $P^{AI}(H_4|\mathbf{x}, \mathbf{y}) = 0.3446$ and $P^{MI}(H_4|\mathbf{x}, \mathbf{y}) = 0.4887$.

Example 2 : The data given below are simulated data of size 10 from Freund's bivariate exponential model with parameters $(\alpha, \alpha', \beta, \beta') = (0.1, 0.3, 0.12, 0.32)$.

i	1	2	3	4	5	6	7	8	9	10
x_i	.0323	.0981	.0216	.4132	1.3709	.0651	.1823	.0086	.0005	.3848
y_i	.2103	.0130	.0990	.1089	.0322	.0155	.0539	.7758	.1906	.0264

For above data, Table 3 indicates Bayes factors and $P(H_1|\mathbf{x},\mathbf{y})$ for testing $H_1 : \alpha = \beta$ v.s. $H_2 : \alpha' \neq \beta'$.

Table 3 : Bayes factors and $P(H_1|\mathbf{x},\mathbf{y})$ for testing $H_1 : \alpha = \beta,\ \alpha' = \beta'$ v.s. $H_2 :$ *not* H_1.

B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
.2985	.2422	.07701	.8050

Also, for above data Table 4 indicates Bayes factors and $P(H_3|\mathbf{x},\mathbf{y})$ for testing $H_3 : \alpha = \alpha',\ \beta = \beta'$ v.s. $H_4 :$ *not* H_3.

Table 4 : Bayes factors and $P(H_3|\mathbf{x},\mathbf{y})$ for testing $H_3 : \alpha = \alpha',\ \beta = \beta'$ v.s. $H_4 :$ *not* H_3.

B_{431}^{AI}	B_{43}^{MI}	$P^{AI}(H_3\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_3\|\mathbf{x},\mathbf{y})$
8.2228	6.6246	0.1084	0.1311

From table 3, since $B_{21}^{AI} = 0.2985$, $P^{AI}(H_1|\mathbf{x},\mathbf{y}) = 0.7701$ and $B_{21}^{MI} = 0.2422$, P^{MI} $(H_1|\mathbf{x},\mathbf{y}) = 0.8050$, there is no strong evidence for H_2 and H_1 in terms of the posterior probability, respectively. That is, there is no strong evidence for non-symmetry for above bivariate data in terms of the posterior probability $P^AI(H_2|\mathbf{x},\mathbf{y}) = 0.2299$ and $P^{MI}(H_2|\mathbf{x},\mathbf{y}) = 0.1950$.

From table 4, since $B_{43}^{AI} = 8.2228$, $P^{AI}(H_3|\mathbf{x},\mathbf{y}) = 0.1084$ and $B_{43}^{MI} = 6.6246$, P^{MI} $(H_3|\mathbf{x},\mathbf{y}) = 0.1311$, there is evidence for H_4 and H_3 in terms of the posterior probability, respectively. That is, there is evidence for independence for above bivariate data in terms of the posterior probability $P^{AI}(H_4|\mathbf{x},\mathbf{y}) = 0.8916$ and $P^{ME}(H_4|\mathbf{x},\mathbf{y}) = 0.8689$.
Therefore, the arithmetic and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

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