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Nonparametric Estimation of Reliability in Time Dependent Strength-Stress Model

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Abstract

We treat the problem of estimating reliability $R(t) = P[Y(t) < X(t)]$ in the time dependent strength-stress model in which a unit of strength $X(t)$ is subjected to environmental stress $Y(t)$ at time t. In this paper two nonparametric approaches to estimate of $R(t)$ are analyzed and compared with parametric method by simulation.

Key Words and Phrases: Stress-Strength Model, Reliability, Wilcoxon-Mann-Whitney Estimator, Kernel Estimator

1. Introduction

A system, whether of a single or multiple components, functions under environmental stress caused by numerous factors. Strength-stress models deal with how long a system functions under stress with how much strength, or how much is the probability that a system functions for a given amount of time. Let Y and X be the stress and strength variables with cdf's $F(x)$ and $G(y)$, respectively. Then, the probability that a system's strength is greather than the stress under which the system is put, i.e. reliability can be calculated as follows;

$$R = P(X < Y) = \int_0^\infty F(u)dG(u) = \int_0^\infty S(u)dF(u)$$

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here, $S(u) = 1 - G(u)$.

Previous work in this field is mostly occupied by the parametric approaches for continuous data. The most general parametric model is the one in which both the strength and the stress are normally distributed. Church and Harris(1970), Downton(1973), Woodward and Kelly(1977) and Reiser and Guttman(1986) have done various analyses in such case. Birnbaum(1956) has proposed the following nonparametric estimator for the R above ;

$$\begin{aligned} \hat{R} &= \int F_m(x)dG_n(x) \\ &= \frac{1}{n}\sum_{i=1}^{n} F_m(Y_i) \\ &= \frac{1}{mn}\{\text{number of } (i,j) \text{ pairs such that } Y_j \geq X_i\}, \end{aligned}$$

where $F_m(\cdot)$ and $G_n(\cdot)$ are empirical cdf's of the X's and Y's respectively.

As briefly stated, the formal studies were about reliability or failure probability of system or components under constant stress independent of time. But, it is easily believed that studies concerning strength-stress models where the stress varies in time should be performed. Until now, works in this direction were few due to various difficulties. After Basu and Ebrahimi(1983), who briefly intoduced $X(t)$, $Y(t)$, the strength and stress variables at time t, this study were not much explored until now. Lee and Kim(1996) studied reliabilities of systems such as computers or TV's where a large amount of stress is given at first when the electricity is turned on, and then decreasing stress in time. Consider the time between the point when the power is turned on till the power is turned off as a cycle. In a cycle, the largest stress will occur when the power is turned on. At this point, if this stress exceeds the fixed strength of the system, this system will cease to function. Otherwise, the system will function until the next cycle. If a breakdown occurs in the k-th cycle, the later cycles will not be of consideration. So, in n number of cycles before time t, the probability that this system will continue to function is equal to the probability that the stress is less than strength in all cycles. Let's assume that $N(t)$, the number of times the power is turned on before time t, is a Poisson process. That is,

$$P[N(t)=n]=\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Then, the reliability $R(t)$ until time t is

$$\begin{aligned}
R(t) &= \sum_{n=0}^{\infty} P\{\text{The event that } n \text{ number of stress occur until time } t\} \\
&\qquad \times P\{\text{The system keeps functioning after } n \text{ stresses}\} \\
&= e^{-\lambda t}\cdot 1+\frac{(\lambda t)e^{-\lambda t}}{1!}\cdot p+\frac{(\lambda t)^2 e^{-\lambda t}}{2!}\cdot p^2+\cdots+\frac{(\lambda t)^n e^{-\lambda t}}{n!}\cdot p^n+\cdots \\
&= e^{-\lambda t}e^{-\lambda p t}=e^{-(1-p)\lambda t}
\end{aligned}$$

where $p = P\{$ The system keeps functioning after one stress $\}$.

We propose a new statistic in section 2, and give properties of it and results of the simulation using this statistic. In section 3, simulation results and conclusion is given.

2. Estimators of $R(t)$

Let $N_1, N_2, \cdots, N_m$ be random variables denoting the number of stress until time t. Assume $N_1, N_2, \cdots, N_m$ are independent and Poisson distributed with mean λt . Also let $X(t_{i_1}), X(t_{i_2}), \cdots, X(t_{i_{N_i}})$ be iid with cdf F, denoting the stress random variables at time $t_{i_j} (i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, N_i)$.

2.1 Wilcoxon-Mann-Whitney type Estimator

In generally, it is well known that if $N_1, N_2, \cdots, N_n$ are independent and Poisson distributed with mean λ, then $\sum_{i=1}^n N_i/n$ is Uniformly Minimum Variance Unbised Estimator(UMVUE) of λ and that if $X_1(t), X_2(t), \cdots, X_n(t)$ be iid with cdf F, then the empirical cdf $\hat{p} = P(\widehat{X(t)} < s) = \sum_{i=1}^n I(X_i(t) < s)/n = F_n(s;t)$ is UMVUE of p for the fixed strength value s.

So, Lee and Kim(1996) proposed the following estimator for $R(t)$ using the empirical cdf for p.

$$\begin{aligned}\widehat{R_{WMW}}(t) &= e^{-(1-\widehat{p_{WMW}})\hat{\lambda}t} \\ &= \exp\left\{-\left(1-\frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{N_i} I(X_{ij}<s)\right)\cdot\frac{N}{m}\right\} \\ &= \exp\left\{-\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{N_i} I(X_{ij}\geq s)\right\} \\ &= \exp\left\{-\frac{1}{m}\sum_{i=1}^{N} I(X_i\geq s)\right\}\end{aligned}$$

where $X_{ij} \equiv X(t_{i_j}),\ N = \sum_{i=1}^{m} N_i(t),$

$$X_1 \equiv X_{11}, X_2 \equiv X_{12}, \cdots, X_{N_1} \equiv X_{1N_1}, X_{N_1+1} \equiv X_{21}, \cdots, X_N \equiv X_{mN_m}.$$

In other words, $\widehat{p_{WMW}} = \sum_{i=1}^{N} I(X_i < s)/N,\ \hat{\lambda} = N/mt$, where $I(X_i < s) = 1$ if $X_i < s$ and 0 otherwise.

Also we showed that $\widehat{R_{WMW}}(t)$ is a consistent estimator for $R(t)$.

2.2 Kernel type estimator

We propose the following estimator, using for the λ a UMVUE type estimator and for the p the Kernel density estimator of.

$$\begin{aligned}\widehat{R_{Ker}}(t) &= e^{-(1-\widehat{p_{Ker}})\hat{\lambda}t} \\ &= \exp\left\{-\left(1-\int_{-\infty}^{s}\frac{1}{Nh}\sum_{i=1}^{N} k\left(\frac{u-X_i}{h}\right)du\right)\cdot\frac{N}{m}\right\}\end{aligned}$$

where $k(\cdot)$ is a kernel function and h is a bandwidth.

The bandwidth is selected by the biased cross-validation method for $\hat{R}$ which is proposed by Scott and Terrell(1987).

Now we show that $\widehat{R_{Ker}}(t)$ is a condistent estimator of $R(t)$.

$$E[\widehat{p_{Ker}}\hat{\lambda}t] = \sum_{n=0}^{\infty} E\left[\int_{-\infty}^{s}\hat{f}(u)du\frac{N}{m}\ |N=n\right]\cdot P(N=n)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} E\left[\int_{-\infty}^{s} \frac{1}{nh}\sum_{i=1}^{n} k\left(\frac{u-X_i}{h}\right) du \frac{n}{m} \; |N=n\right] \cdot P(N=n) \\
&= \sum_{n=0}^{\infty} \frac{n}{m}\frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{s} k\left(\frac{u-y}{h}\right) f(y)dudy \cdot P(N=n) \\
&= \sum_{n=0}^{\infty} \frac{n}{m}\int_{-\infty}^{\infty} K\left(\frac{s-y}{h}\right) f(y)dy \cdot P(N=n) \\
&\longrightarrow \sum_{n=0}^{\infty} \frac{n}{m}\int_{-\infty}^{\infty} I(s-y)f(y)dy \cdot P(N=n) \quad \text{as } h \to 0 \\
&= \sum_{n=0}^{\infty} \frac{n}{m} p \cdot P(N=n) = p\lambda t
\end{aligned}
$$

, where $f(\cdot)$ is the pdf of $F(\cdot)$. Here $I(x) = 1$ if $x > 0$ and 0 otherwise, and $K(\cdot)$ is the cumulative distribution function of $k(\cdot)$. Therefore $E[(1-\widehat{p_{Ker}})\hat{\lambda}t] = (1-p)\lambda t$. So $(1-\widehat{p_{Ker}})\hat{\lambda}t$ is asymptotically unbiased.

$$
\begin{aligned}
Var[(1-\widehat{p_{Ker}})\hat{\lambda}t] &= Var\left(\int_{s}^{\infty} \hat{f}(u)du\frac{N}{m}\right) \\
&= \frac{1}{m^2}\Bigg[Var\left\{E\left(\frac{1}{Nh}\sum_{i=1}^{N}\int_{s}^{\infty} k\left(\frac{u-X_i}{h}\right) du \; N \,\Big|N\right)\right\} \\
&\qquad +E\left\{Var\left(\frac{1}{Nh}\sum_{i=1}^{N}\int_{s}^{\infty} k\left(\frac{u-X_i}{h}\right) du \; N \,\Big|N\right)\right\}\Bigg] \\
&= \frac{1}{m^2}\Bigg[Var\left\{\frac{N}{h}\int_{-\infty}^{\infty}\int_{s}^{\infty} k\left(\frac{u-X_i}{h}\right) du f(y)dy\right\} \\
&\qquad +E\left\{N^2\frac{1}{N}Var\left(\frac{1}{h}\int_{s}^{\infty} k\left(\frac{u-X_i}{h}\right) du\right)\right\}\Bigg] \\
&\longrightarrow \frac{1}{m^2}\{p^2 m\lambda t \; + \; p(1-p)m\lambda t\} \quad \text{as } h \to 0 \\
&= \frac{1}{m^2} pm\lambda t \; \to 0 \;\; \text{as } m \to \infty.
\end{aligned}
$$

This shows that since $(1-\widehat{p_{Ker}})\hat{\lambda}t$ satisfies asymptotic unbiasedness, it is a consistent estimator for $(1-p)\lambda t$ as $m \to \infty$ and $h \to 0$. So it follows that $\widehat{R_{Ker}}(t)$ is a consistent estimator of $R(t)$.

2.3 Case of known stress distribution G

In some applications, the stress distribution may be known to the investigator. The following is an example of situations in which the stress distribution may be known. Assume the parametric model $Y \sim G(\cdot|\theta)$ for stress. Instead of $\widehat{p_{MLE}} = P(\widehat{X < s}|\theta)$, we use $G(s|\hat{\theta})$ by caculating the maximum likelihood estimator(M.L.E. : $\hat{\theta}$) of θ. For example, when $Y \sim N(\mu, \sigma^2)$, let $G(s\ |\hat{\theta}) = \Phi\left((s-\hat{\mu})/\hat{\sigma}\right)$ where $\hat{\mu}$ and $\hat{\sigma}$ are M.L.E. of μ and σ, respectively. Then

$$\widehat{R_{MLE}}(t) = \exp\Big(-(1-G(s|\hat{\theta}))\hat{\lambda}t\Big)$$

Now we discuss the consistency of $\widehat{R_{MLE}}(t)$.

Under some regularity conditions on G (see Lehmann (1983), P409, for example), the MLE $\hat{\theta}$ is a consistent estimator of θ . Suppose, in addition, that $\sup_{|\theta-\theta'|\le c}\left|\frac{\partial}{\partial\theta}G(y|\theta')\right| \le M(y)$ for all y in the support of $G(\cdot|\theta)$ with $E(M(Y)) < \infty$ and $c > 0$. By Taylor expansion, we have

$$G(s\ |\hat{\theta}) = G(s\ |\theta) + (\hat{\theta}-\theta)^T \cdot \frac{\partial}{\partial\hat{\theta}}G(s\ |\theta^*)$$

where θ^* is a point on the line segment joining $\hat{\theta}$ and θ.

Thus

$$\begin{aligned}\Big|G(s\ |\hat{\theta}) - G(s\ |\theta)\Big| &= \Big|(\hat{\theta}-\theta)^T \cdot \frac{\partial}{\partial\hat{\theta}}G(s\ |\theta^*)\Big| \\ &= \Big|\hat{\theta}-\theta\Big| \cdot M(s) \\ &\to 0, \quad \text{in probability.}\end{aligned}$$

This show that it is a consistent estimator of $R(t)$.

3. Simulation Results

In order to get the performance of estimated values of $R(t)$ and their MSE's(Mean Squared Error), computer simulations using FORTRAN IMSL (International Mathematical and Statistical Library), were performed. In this simulations, the distribution of stress was taken as Weibull, in which the scale parameter was fixed as 2.0

and the shape parameter took varying valus 0.5, 1.0, 3.0 and 3.5. The stress value was fixed at 3.0, the time t was changed from the point at which $R(t)$ took value 0.05 to the value 0.95. This process was repeated 3,000 times, calculating the MSE.

The result of the simulation study is as follows. The MSE was shown to be in the increasing from when the value of $R(t)$ was in the range of 0.05   0.5, and decreasing from in the range of 0.5   0.99. For the interval where $0.1 < R \leq 0.9$, MLE method is always the best and KER method is best in the tail part, $R \leq 0.1$ or $R > 0.9$. The MSE's of the nonparametric approach are some unstable where the value of the shape parameter is big. Looking from the MSE point, the Kernel method yielded the least MSE and thus is favorable in the nonparametric approach, but its simulation time was very long compared to the other methods.

The following figures 1   4 shows the changing level of MSE as the value of the shape parameter of the Weibull distribution varies.

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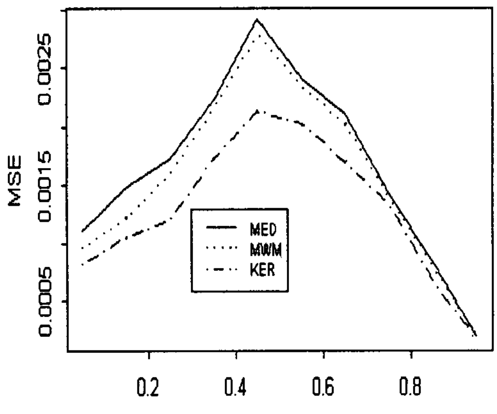

Figure 1. Weibull Case,(Shape Para=0.5, Scale para=2.0)

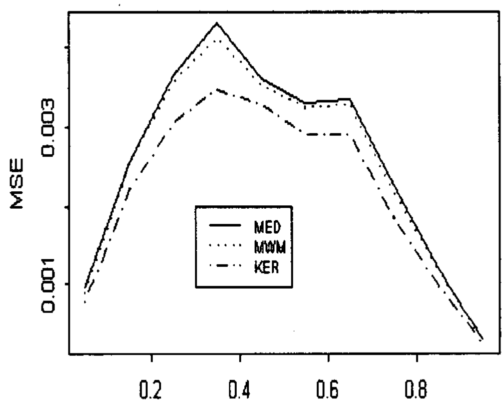

Figure 2. Weibull Case,(Shape Para=1.0, Scale para=2.0)

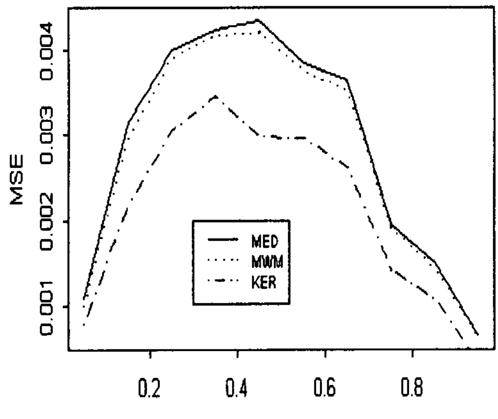

Figure 3. Weibull Case,(Shape Para = 3.0, Scale para=2.0)

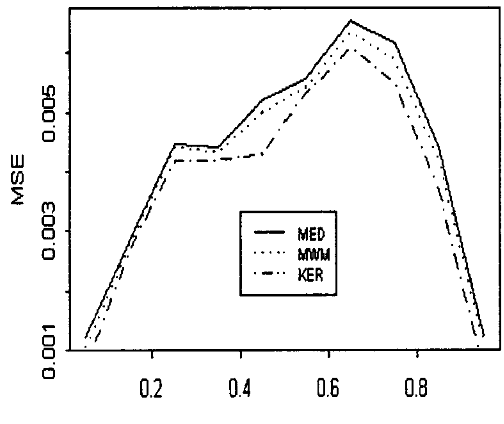

Figure 4. Weibull Case,(Shape Para = 4.0, Scale para=2.0)