Journal of the Korean
Data & Information Science Society
1999, Vol. 10, No. 1, pp. 67 ~ 77

A Test Based on Euler Angles of a Rotationally Symmetric Spherical Distribution [1]

Yang-Kyu Shin [2]

Abstract

For a orientation-shift model supported on the unit sphere, Euler angles are the conventional measure to parametrize orientation-shifts. The essential role which is played by rotationally symmetry of an underlying distribution is reviewed. In this paper we propose the inference procedure based on Euler angles for the rotationally symmetric spherical distribution. The likelihood ratio test(LRT) based on the Euler angles is worked out. The asymptotic distribution of the test under the null hypotheses and certain contiguous alternatives is obtained.

Key words and Phrases: Orientation-shift, Euler-angles, Fisher distribution, Dimroth-Watson distribution, LRT.

1. Introduction

Various statistical inference problems have been formulated to deal with spherical data. As a result, many spherical distributions as well as various inference techniques, parametric or nonparametric, has been proposed (Mardia(1975) and Fisher, Lewis and Embleton(1993)). On the parametric side, though well-known

[1]This research was supported in part by Korea Institute of Oriental Medicine Grant for 1998
[2]Associate Professor, Dept. of Information Stitistics, Kyungsan University, Kyungpook, 712-240, Korea

distributions such as Fisher distribution or Bingham distribution have attracted frequent attention(Bingham(1974) and Best and Fisher(1984, 1986)), there have been relatively few attempts to develop general inference procedures, either scale or location, for a large class of spherical distributions. Watson(1983) introduced the eigenvalues of the sample second moment matrix to classify different rotational symmetries. When there is enough symmetry, by Euler angles Θ, proper rotations are parametrized. The Fisher distribution and the Dimroth-Watson distribution are rotationally symmetric about the mean vector and an appropriate eigenvector of the second moment matrix of the Dimroth-Watson distribution, respectively. In this paper, we develop inference procedure based on Euler angles for a particular type of density which includes as special cases, the Fisher distribution and the Dimroth-Watson distribution.

In section 2, the role which is played by rotational symmetry of an underlying distribution is discussed and an orientation-shift model for the unit random vectors in R^3 is reviewed(Kim(1978)) to furnish the necessary background. In section 3, inference procedure based on Euler angles are worked out. Let Y be spherical random vector with a probability density function g. By employing an analogy between the random vector Y with a probability density function g and the unit spherical shell with the same mass density function, an orientation-shift from Y to ΓY by a proper rotation $\Gamma \in O^+(3)$, where $O^+(3)$ is the group of proper rotations, or the group of 3×3 orthogonal matrices with determinant $+1$, can be viewed as the equivalent rotation of the spherical shell. Euler angles Θ are introduced to parameterize proper rotations. And the asymptotic distributions of a moment estimate of the Euler angles are obtained. Finally the asymptotic distribution of 1-step estimate of Le Cam under the null hypothesis $H_0 : \Theta = \Theta_0$ and contiguous alternatives $K_n : \Theta = \Theta_0 + \frac{h}{\sqrt{n}}, h \in R^3$, are derived; asymptotic optimalities are considered for the test.

2. Background

2.1 Models for spherical distribution

Let λ be the Lebesgue measure on $S^2 = \{x \, ; \, \|x\| = 1, \ x \in R^3\}$, the unit sphere in

R^3 ; $d\lambda\ (\theta,\ \phi) = \sin\theta d\theta d\phi$, where $(\theta,\ \phi\ )$ are the polar angles of $x\ =\ (x_1, x_2, x_3) \in S^2$ so that

$$x_1 = \sin\theta\cos\phi,\ x_2 = \sin\theta\sin\phi,\ x_3 = \cos\theta.$$

Among many spherical distributions proposed in the literature (Mardia(1972), Watson(1983) and Fisher, Lewis and Embleton(1993)), Fisher distribution, and Dimroth-Watson distribution are well-known. A 3-dimensional spherical vector x is said to have a Fisher distribution if its probability density function is given by $f(x) = a(k)\exp(k\mu' x),\ (k > 0,\ \mu \in S^2), x \in S^2$, where k is called the concentration parameter and $a(k)$ is the normalizing constant. Clearly, the distribution is rotationally symmetric about μ which is the mean vector of the distribution. Owing to the rotational symmetry, the probability density function, can be concisely written as

$$f(\theta, \phi) = a(k)\ \exp(k\cos\theta)\sin\theta\ ,\ 0 < \theta < \pi,\ 0 < \phi < 2\pi,$$

where θ is the angle between x and the mean direction vector μ. The Fisher distribution is analogue on the sphere S^2 to the isotropic bivariate normal distribution in the plane. A spherical vector x has the Bingham distribution if its probability density function is given by

$$f(x) = d(K)\exp(tr(KMxx\prime M)),\ x \in S^2,$$

where K is a 3×3 diagonal matrix with constants k_1, k_2, and k_3, and M is a 3×3 orthogonal matrix, and $d(K)$ is the normalizing constant depending only on K. It may be noted that the dependence of f on K is unique up to adding a constant to k_1, k_2, and k_3. The probability density function can be explicitly written as

$$f(x) = d(K)\exp\left\{k_1(\mu_1\prime x)^2 + k_2(\mu_2\prime x)^2 + k_3(\mu_3\prime x)^2\right\}, \qquad (1)$$

where $K = (k_1, k_2, k_3)$ and $\{\mu_1,\ \mu_2,\ \mu_3\}$ is orthonormal set. The Bingham distribution is appropriate as a model for axial data because of the antipodal symmetric property: $f(x) = f(-x)$. Depending on the values of K in (1), we have different types of spherical distribution:

i) $k_1 = k_2 = k_3$; The uniform distribution.

ii) $k_1 = k_2 = 0,\ k_3 > 0$; A symmetric axial distribution about μ_3.

iii) $k_1 = k_2 = 0,\ k_3 < 0$; A symmetric girdle distribution with μ_3 as the axes of symmetry.

iv) $k_1 > k_2 > k_3 = 0$; An asymmetric axial distribution about the plane normal to μ_1.

v) $k_1 < k_2 < k_3 = 0$; An asymmetric girdle distribution about the plane normal to μ_1.

For our purpose, in modelling, we restrict our attention to rotationally symmetric distribution through this paper; hence, for Bingham distribution, only the cases (ii) and (iii) will be considered which are called the Dimroth- Watson distribution.

2.2 Orientation-shift Model

Suppose a sample is taken on a unit random vector X with probability density function of the form

$$f(x) = g(\Gamma^{-1}x), x \in S^2, \Gamma \in O^+(3),$$

where the orientation-shift Γ from g to f is to be estimated from the sample. Let Y be a unit random vector with probability density function g. Define

$$Y_R = \{\Gamma Y : \Gamma \in O^+(3)\}.$$

Similarly, noting that the probability density function of ΓY is $g \circ \Gamma^{-1}$, define

$$g_R = \{g \circ \Gamma^{-1} : \Gamma \in O^+(3)\}.$$

Then Y_R is the orientation family generated by Y, and g_R is the orientation family generated by g. Not all the members of g_R are distinct in the presence of an underlying density, inducing natural equivalence relations on $O^+(3)$ and in Y_R :

$$\Gamma_1 \sim \Gamma_2 \text{ if and only if } g \circ \Gamma_1^{-1} = g \circ \Gamma_2^{-1};$$

$$\Gamma_1 Y \sim \Gamma_2 Y \text{ if and only if } \Gamma_1 = \Gamma_2.$$

Then a proper rotation Γ is the orientation-shift from Y (or g) to ΓY (or $g \circ \Gamma^{-1}$). This leads to the following description : an orientation-shift model consists of a reference vector Y, the group of orientation-shifts $O^+(3)$, the orientation family Y_R generated by Y through the orientation-shifts in $O^+(3)$, and the equivalence classes in Y_R induced by the underlying symmetry. In considering a sample from the common density $g \circ \Gamma^{-1}$ with Γ to be estimated, it is now clear that $\bar{\Gamma}$ rather than the particular member Γ is what needs to be estimated; it is enough to find an estimate for any $\Gamma_1 \sim \Gamma$.

To discuss how to recover $\bar{\Gamma}$, we first need some notations. Define

$$\mu_Y = E(Y), \mu_{\Gamma Y} = E(\Gamma Y) \text{ and } M_Y = E(YY'), M_{\Gamma Y} = E[(\Gamma Y)(\Gamma Y)']. \quad (2)$$

Let $\sum(Y)$ (and $\sum(\Gamma Y)$) be an orthogonal matrix whose columns are the oriented, ordered and orthonormalized eigenvectors of the second moment matrix M_Y (and $M_{\Gamma Y}$).

Define that Y (or the distribution of Y) is $\mu(0)$ if $\mu_Y = 0; Y$ is $\mu(j)(j = 1, 2, 3)$ if $\mu_Y \neq 0$ and $\mu_Y X \sigma_{jY} = 0$ for some nondegenerate eigenvector σ_{jY}.

Also Define that Y (or the distribution of Y) is $D(\alpha)$ if α is the largest multiplicity of the eigenvalues of M_Y. In particular we say Y is $D(2 : j)$ to emphasize that the j-th eigenvalue is non-degenerate. Finally define that the subclass ${}_\beta C_\alpha$ of C is the class of spherical distributions satisfying condition which $D(\alpha)$ and $\mu(\beta)$.

With canonically selected Y, $\sum(Y)$ and $\sum(\Gamma Y)$, the shift $\bar{\Gamma}$ from Y to ΓY is recovered by

$$\Gamma_1 e_3 = \sigma_{3\Gamma Y}, \ \Gamma_1 \in O^+(3), \quad (3)$$

where $\sigma_{3\Gamma Y}$ satisties $\mu_Y \cdot e_3 = \mu_{\Gamma Y} \cdot \sigma_{3\Gamma Y}$ and e_3 denotes the third coordinate vector.

Let $X_1, X_2, \cdots, X_n$ be independent observations on $X = \Gamma Y$ for some unknown $\Gamma \in O^+(3)$, where $Y \in C$ is canonically seleted. Then we take the canonically selected

$$\widehat{\Gamma}_n e_3 = \widehat{\sigma}_{3n}$$

where the selection $\widehat{\sigma}_{3n}$ is arbitrary for ${}_oC_{2:3}$ and satisfies $(\mu \cdot e_3)(\widehat{\mu}_n \cdot \widehat{e}_{3n}) > 0$ for ${}_3C_{2:3}$, as our estimator of $\bar{\Gamma}$.

2.3 Euler-angles on the Sphere

Since the columns of orthogonal matrices satisfy three orthogonality conditions between themselves and three more normality constraints on each column, $O^+(3)$ can be parametrized by a suitable set of three independent parameters. The set of Euler angles can used to describe rotational motions of rigid bodies. Any proper rotation Λ can be performed as a succession of three elementary rotations about the coordinate vectors by Euler angles. They consist of : the first rotation which is made about the e_3-axis by an angle ϕ, described by $R_3(\phi)$; the second rotation about the e_2-axis by an angle θ, described by $R_2(\theta)$; the third and final rotation about the e_3-axis by an angle ψ, described by $R_3(\psi)$. The complete rotation is given by $\Gamma(\Theta) = R_3(\psi)R_2(\theta)R_3(\phi)$ with $\Theta = (\phi, \theta, \psi)'$:

$$\begin{aligned}\Gamma(\Theta) &= R_3(\psi)R_2(\theta)R_3(\phi) \\ &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi & \sin\theta\cos\psi \\ \cos\phi\cos\theta\sin\psi + \sin\phi\sin\psi & -\sin\phi\cos\theta\cos\psi + \cos\phi\cos\psi & \sin\phi\sin\psi \\ -\cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \end{bmatrix}\end{aligned}$$

3. Testing

In this section, we develop the inference procedure based on Euler angles to a rotationally symmetric family generated by the density of the form

$$f(x) = a(k)h[k(I - v\cdot x)], k \neq 0, v, x \in S^2, \tag{4}$$

where $h(\neq constant)$ is assumed to satisfy the regularity conditions. Both the Fisher distribution and the Dimroth-Watson distribution have densities of this form. To choose the canonical reference density for the family $f_R = \{f \circ \Gamma^{-1} : \Gamma \in O^+(3)\}$, we first need to classify the distribution into one of the subclasses ${}_\beta C_\alpha$ of C. Let X be the random vector with density f. Since f is circularly symmetric about v, if $\mu_X = 0$, the v-axis concides with the nondegenerate eigenaxis of M_X and X is ${}_0C_{2:j}$ with $j = 1$ or 3 depending on h and k. If $\mu_X \neq 0$, the v-axis concides with both the mean axis and the nondegenerate eigenaxis, and X is ${}_jC_{2:j}$. Let us assume that $j = 3$ in either case. Now according to the canonical selection, the canonical reference vector Y has the canonical density g given by, upon setting $v = e_3$ in (4),

$$g(y) = a(k)h[k(1-y_3)] . \tag{5}$$

Let $X_1, X_2, \cdots, X_n$ be a sample from the common density f. To estimate the shift $\Gamma \in 0^+(3)$ (up to the equivalence induced by f) from Y to $X = \Gamma Y$, or equivalently, from e_3 to $v = \Gamma e_3$, we simply put $\widehat{\sigma}_{3n} = \widehat{\Gamma}_n e_3, \widehat{\Gamma}_n \in 0^+(3)$, where the canonical selection $\widehat{\sigma}_{3n}$ satisfies $(u_Y \cdot e_3)(\widehat{u}_n \cdot \widehat{\sigma}_{3n}) \geq 0$. And $\widehat{\theta}_n = (\widehat{\theta}_n, \widehat{\psi}_n)'$ is obtained by setting $R_3(\widehat{\psi}_n)R_2(\widehat{\theta}_n) = \widehat{\sigma}_{3n}$, i.e., by the polar angles of $\widehat{\sigma}_{3n}$; for any ϕ, $\widehat{\Gamma}_n = R_3(\widehat{\psi}_n)R_2(\widehat{\theta}_n)R_3(\phi)$ qualifies as an estimate of $\widehat{\Gamma}$.

The asymptotic distribution of $(\widehat{\theta}_n, \widehat{\psi}_n)'$ is given by the following Lemma 1.

Lemma 1. Let $(\theta_1, \psi_1)'$ and $(\widehat{\theta}_n, \widehat{\psi}_n)'$ be the polar angles of v and the canonical choice $\widehat{\sigma}_{3n}$, respectively. Assuming that $\theta_1 \neq 0$ or π, we have

$$[(\sqrt{n}\ (\widehat{\theta}_n - \theta_1), \sqrt{n}\ (\widehat{\psi}_n - \psi_1))'|(\theta_1, \psi_1)'] \sim N\left[0, \frac{\alpha_1}{(d_3-d)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta_1 \end{pmatrix}\right],$$

where $d_3 = 1 - 2d = 2\pi\int_{-1}^{1} y^2 g_3(y)dy$ with $g_3(y) = a(k)h[k(1-y)]$ and $\alpha_1 = \pi\int_{-1}^{1}(1-y^2)y^2 g_3(y)dy$.

Proof. Since $\sigma_{3n} = v$, $[\sqrt{n}(\widehat{\theta}_n - \theta_1, \sqrt{n}(\widehat{\psi}_n - \psi_1)]' \to -(d_3-d)^{-1}As$, where

$$A = \begin{bmatrix} -\frac{v_2}{\sqrt{1-v_3{}^2}} & \frac{v_1}{\sqrt{1-v_3{}^2}} \\ -\frac{v_1}{1-v_3{}^2} & \frac{v_2}{1-v_3{}^2} \end{bmatrix}$$

with $v = (v_1, v_2, v_3)'$, and $s = (-s_{23}, s_{31})'$. Note that $1 - v_3^2 = \sin^2\theta_1 \neq 0$ by the assumption. By the circular symmetry of g, s is seen to be distributed as $N(0,\ \alpha_1 I_{2\times 2})$.

Hence, direct computation yields, since $\|v\| = 1$,

$$-(d_3-d)^{-1}As \sim N(0, \frac{\alpha_1}{(d_3-d)^2}AA') = N\left[0, \frac{\alpha_1}{(d_3-d)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta_1 \end{pmatrix}\right].$$

The expression of the limiting covariance matrix is worth noting in that, being sensible estimates of θ and ψ, $\widehat{\theta}_n$ and $\widehat{\psi}_n$ are asymptotically independent and the variance of $\widehat{\psi}_n$ approaches ∞ as $\theta_1 \to 0$.

To improve $(\widehat{\theta}_n, \widehat{\psi}_n)'$ Le Cam's 1-step estimator $(\theta_n^*, \psi_n^*)'$ defined by

$$\Theta_n^* = \widehat{\Theta}_n + \frac{1}{\sqrt{n}} I^{-1}(\widehat{\Theta}_n)\Delta_n(\widehat{\Theta}_n),$$

where $I(\Theta) = E[h(\Theta)h(\Theta)'/h(\Theta)^2]$ with $h(\Theta) = g[\Gamma(\Theta)^{-1}X]$, $h(\Theta) = \frac{d}{d\Theta}h(\Theta)$ and $\sqrt{n}\Delta_n(\Theta) = \sum_{j=1}^n \frac{d}{d(\Theta)}\log g\ [\Gamma(\Theta)^{-1}X_j]$.

Let g be the reference density of (4). Then $\frac{d}{d\theta}g(\Gamma(\Theta)^{-1}X) = V_3's(Y)$, where $\Theta = (\theta, \psi)'$, $Y = \Gamma(\Theta)^{-1}X = R_2(-\theta)R_3(-\psi)X$,

$$V_3' = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \text{and } s(Y) = Y \times \frac{d}{dY}\log g(Y).$$

The expression for $(\theta_n^*, \psi_n^*)'$ is derived in the following.

Lemma 2. Suppose $\theta_1 \neq 0$ or π. Then, with the same notations as in Lemma 1,

$$(\theta_n^* - \widehat{\theta}_n, \psi_n^* - \widehat{\psi}_n)' = -\frac{1}{\alpha^2}(\widehat{\beta}_{1n} \cdot \widehat{\mu}_n, \frac{1}{\sin\widehat{\theta}_n}\widehat{\beta}_{2n} \cdot \widehat{\mu}_n)',$$

where

$$\begin{aligned} \widehat{\beta}_{1n} &= (\cos\widehat{\theta}_n \cos\widehat{\psi}_n, \cos\widehat{\theta}_n \sin\widehat{\psi}_n, -\sin\widehat{\theta}_n)', \\ \widehat{\beta}_{2n} &= (-\sin\widehat{\psi}_n, \cos\widehat{\psi}_n, 0)', \\ \widehat{\mu}_n &= \frac{1}{n}\sum_{j=1}^n w(\widehat{\sigma}_{3n} \cdot X_j)X_j \end{aligned}$$

with $w(y) = \frac{d}{dy}\log g_3(y)$, and $\alpha_2 = \pi \int_{-1}^1 (1-y^2)w^2(y)g_3(y)dy$.

Proof. Since $g_3(y_3) = g(y)$, we have $s(Y) = Y \times \frac{d}{dy}\log g_3(Y_3) = (Y_3 w(Y_3), Y_1 w(Y_3), 0)'$. This and the circular symmetry of Y yield

$$I_g = E_Y[s(Y)s(Y)'] \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

where $\alpha_2 = \pi \int_{-1}^{1}(1-y^2)w^2(y)g_3(y)dy > 0$. From V_3 and (5), we have,

$$I(\widehat{\theta}_n, \widehat{\psi}_n) = \widehat{V}_{3,n}' I_g \widehat{V}_{3,n} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \sin^2 \widehat{\theta}_n \end{bmatrix},$$

where $\widehat{V}_{3,n} = V_3|_{(\theta,\psi)=(\widehat{\theta}_n,\widehat{\psi}_n)}$. Hence we have

$$I^{-1}(\widehat{\theta}_n, \widehat{\psi}_n) = \frac{1}{\alpha_2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2 \widehat{\theta}_n} \end{bmatrix}. \tag{7}$$

By Le Cam(1956),

$$\Theta_n^* = \widehat{\Theta}_n + \frac{1}{n} \widehat{V_n}^{-1} I_g^{-1} \sum_{j=1}^{n} S(\widehat{Y}_{jn}).$$

Therefore

$$(\theta_n^* - \widehat{\theta}_n, \psi_n^* - \widehat{\psi}_n)' = -\frac{1}{\alpha^2}(\widehat{\beta}_{1n} \cdot \widehat{\mu}_n, \frac{1}{\sin \widehat{\theta}_n} \widehat{\beta}_{2n} \cdot \widehat{\mu}_n)',$$

where

$$\begin{aligned} \widehat{\beta}_{1n} &= (\cos \widehat{\theta}_n \cos \widehat{\psi}_n, \cos \widehat{\theta}_n \sin \widehat{\psi}_n, -\sin \widehat{\theta}_n)', \\ \widehat{\beta}_{2n} &= (-\sin \widehat{\psi}_n, \cos \widehat{\psi}_n, 0)', \\ \widehat{\mu}_n &= \frac{1}{n} \sum_{j=1}^{n} w(\widehat{\sigma}_{3n} \cdot X_j) X_j \end{aligned}$$

with $w(y) = \frac{d}{dy}\log g_3(y)$, and $\alpha_2 = \pi \int_{-1}^{1}(1-y^2)w^2(y)g_3(y)dy$.

Let us review the lemma for two special distributions, the Fisher distribution and the Dimroth-Watson distribution. When g is the density of the Fisher distribution with $g_3(y) = c(k)l^{ky}$ $(k > 0)$, $w(y) = k$ and $\widehat{\mu}_n$ smplies to $\widehat{\mu}_n = \frac{k}{n}\sum_{j=1}^{n} X_j = k\widehat{\mu}_n$, a multiple of the sample mean vector. When g is the density of the Dimroth-Watson distribution with $g_3(y) = c(k)l^{ky^2}$ $(k \neq 0)$, we expect the asymptotic distributions of $(\widehat{\theta}_n, \widehat{\psi}_n)'$ and $(\theta_n^*, \psi_n^*)'$ to be the same. To check this, since the asymptotic distribution of $[\sqrt{n}(\theta_n^* - \theta_1),\ \sqrt{n}(\psi_n^* - \psi_1)]'$ is $N[O,\ I^{-1}(\theta_1, \psi_1)]$ while that of $[\sqrt{n}(\widehat{\theta}_n - \theta_1),\ \sqrt{n}(\widehat{\psi}_n - \psi_1)]'$ is given above by Lemma 1, we only need to show the equality $(d_3 - d)^2 = \alpha_1\alpha_2$ as indicated by comparing (7) to Lemma 1. After some tedious calculations, we get $\alpha_1 = d_3 - \frac{d}{2k}$ which with $w(y) = 2ky$ confirms the equality.

In the case of 1-sample testing $H_0 : (\theta, \psi)' = (\theta_0, \psi_0)'(\theta_0 \neq 0 \text{ or } \pi)$, the test statistic T_n is given by

$$\begin{aligned} T_n &= \sum_{j=1}^{n} \left\{ \log g[R_2(-\theta_n^*)R_3(-\psi_n^*)X_j - \log g[R_2(-\theta_0)R_3(-\psi_0)X_j]^2 \right\} \\ &= \sum_{j=1}^{n} \left\{ \log h[k(1 - R_3(\psi_n^*)R_2(\theta_n^*)e_3 \cdot X_j)] - \log h[k(1 - R_3(\psi_0)R_2(\theta_0)e_3 \cdot X_j)]^2 \right\} \\ &= \sum_{j=1}^{n} \log h[k(1 - X_{1j}\sin\theta_n^*\cos\psi_n^* - X_{2j}\sin\theta_n^*\sin\psi_n^* - X_{3j}\cos\theta_n^*)] \\ &\qquad - \log h[k(1 - X_{1j}\sin\theta_0\cos\psi_0 - X_{2j}\sin\theta_0\sin\psi_0 - X_{3j}\cos\theta_0)]^2 \end{aligned}$$

with $X_j = (X_{1j}, X_{2j}, X_{3j})'$.

Since the asymptotic distribution of T_n is X_2^2 under H_0, the critical value c_α at a given level α can be read from the table. For a fixed sample size n_0, the approximate power of the test $I_{[T_{n_0} > c_\alpha]}$ at $(\theta_1, \psi_1) \neq (\theta_0, \psi_0)$ can be computed by embedding (θ_1, ψ_1) in $\{K_n\}$, i.e., by taking $h_0 = [\sqrt{n_0}(\theta_1,\ \theta_0),\ \sqrt{n_0}(\psi_1, \psi_0)]'$; T_{n_0} is then distributed approximately as $X_2^2(h_0' I(\theta_0, \psi_0)h_0)$.

Reference

1. Best, D. J. and Fisher, N. I. (1986). Goodness-of-fit and discordancy tests for samples from the Watson distribution on the sphere. *Austral. J. Statist.*, 28, 13-31.

2. Bingham, C. (1974). An antipodally symmetric distribution on the sphere. *Ann. Statist.*, 2, 1201-1225.

3. Fisher, N. I. and Best, D. J. (1984). Goodness-of-fit tests for Fisher's distribution on the sphere. *Austral. J. Statist.*, 6, 142-150.

4. Fisher, N. I., Lewis, T. and Embleton, B. J. J. (1993). *Statistical analysis of spherical data.* Cambridge Univ. Press, Cambridge.

5. Kim, K. M. (1978). *Orientation-shift Model on the Sphere.* Ph.D. thesis. Univ. of California, Berkery.

6. Mardia, K. V. (1975). Statistics of directional data(with discussion). *J. R. Statist. Soc. B*, 37, 349-393.

7. Watson, G. S. (1983). *Statistics on Spheres.* John Wiley and Sons, New York.