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The Estimation of MRLF for Whole Line on LTRC Model

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Abstract

In this paper, for whole line (0, ∞), the estimation procedure of mean residual life function using product-limit estimator is studied with asymptotic properties. And also, the small sample properties of proposed estimator of MRLF are investigated through Monte Carlo study and compared with Kaplan-Meier type estimator.

Key Words and Phrases: mean residual life function, product limit estimator, lefr and right censoring model, whole line.

1. Introduction

The estimation procedure of the mean residual life function(MRLF) has been placed on important role in the area of survival analysis since the MRLF $e(t)$ is represented as the function of survival function $E\ (\ X - t \mid X\ >\ t\ )$ (Cox(1961) and Swartz(1973)). And also, the estimators of MRLF using the parametric and nonparametric methods have been proposed and studied by researchers.

In a random censoring model, Yang(1977, 1978) has proposed the estimator of MRLF and showed the asymptotic properties in a given finite interval. Using the result of Yang, Kumazawa(1987) has proposed the estimator of MRLF using Kaplan-Meir(PL1) estimator(1958) and showed the asymptotic properties on whole line (0, ∞).

Especially, Wang, Jewell and Tsai(1986) considered the left truncated and right censoring model(LTRC) and proposed the produce limit estimator of survival function. And also, they compared with PL1 estimator ignoring left truncation effects and showed PL1 estimator overestimates the survival function than Product-Limit(PL2) estimator.

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In this paper, the estimator of MRLF using the PL2 estimator on LTRC is proposed and studied asymptotic properties on whole line $(0, \infty)$. And also, the small sample properties are studied through the Monte Carlo study.

2. Preliminaries and proposed Estimator

For the estimator of MRLF on LTRC, some useful notations and lemmas are given here.

In LTRC model, let $X_i, \quad i = 1, 2, \cdots, n$ be the lifetime following the distribution F and ,T_i, C_i be the random truncation time and the censoring time with distibution function G and H, respectively to be independent of $X_i, \quad i = 1, 2, \cdots, n$. Suppose that $(Y_i,\ \delta_i,\ T_i\ )$ is observable only when $X_i \geq T_i$, where $Y_i = min(X_i,\ C_i\ )$ and

$$\delta_i = \begin{cases} 1 & \text{if } X_i \leq C_i \\ o & \text{if } X_i \geq C_i \end{cases}$$

Thus, the observed data is given by a set of n independent identically distubuted observations $(Y_i,\ \delta_i, T_i\ ),\quad i = 1, 2, \cdots, n$.

Using above model, Tsai, Jewell and Wang(1987) suggested the product-limit estimator for the survival function $S_{PL2}(t)$ given by

$$\hat{S}_{PL2}(t) = \sum_{x_j \leq t} \frac{n_j - d_j}{n_j}$$

with d_j is the number of failures at time x_j and $n_j = \sum I\ (t_i \leq x_j \leq y_i\ )$ is the number in the risk set at time x_j, where I is the usual indicator function. Note that $\hat{S}_{PL2}(t)$ reduces to the PL1 estimator for right censored data if $T_1 = T_2 = \cdots = T_n = 0$.

For the estimation of $S(t)$ in a given interval, $t \in [0,\ K]$, let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of $X_1, X_2, \cdots, X_n$ and $s_{(1)}, s_{(2)}, \cdots, s_{(r)}$ among $X_1, X_2, \cdots, X_n$ be the distinct observed lifetimes.

It can be easily seen that the estimator of $S(t)$ is the step function with possible jumps at $s_{(1)}, s_{(2)}, \cdots, s_{(r)}$. Then Tsai, Jewell and Wang(1987) suggested the nonparmetric maximum likelihood estimator for the survival function $S(t)$ given by

$$\hat{S}_{PL2}(t) = \prod_{i\ :\ S_{(i)} \leq t} \left(1 - \frac{\Delta N(s_{(i)})}{Y(s_{(i)})}\right)$$

with $N(t) = \sum_{i=1}^{n} I\ (\ X_i \leq t,\ \delta_i\ =\ 1\ )$ and $Y(t)\ =\ \sum_{i=1}^{n}\ I\ (\ T_i\ \leq\ t \leq\ X_i\ )$.

Tasi, Jewell and Wang(1987) showed the consistency and the weak convergency of the product-limit estimator $\hat{S}_{PL2}(t)$ as follows.

Lemma 2.1 (Tasi, Jewell and Wang(1987)) Let F, G and H be continuous. Then for $0 \le t \le K$,
a) $Pr(|\hat{S}_{PL2}(t) - S(t)|) \rightarrow 0$ with probability 1 as $n \rightarrow \infty$.
b) $\sqrt{n}$ ($|\hat{S}_{PL2}(t) - S(t)|$) $\rightarrow$ $Z^*(t)$ weakly in D [0, K], where $Z^*(t)$ is a mean zero Gaussian process with covariance structure given by

$$Cov(Z^*(s)\ ,\ Z^*(t)\ )\ =\ S(s)S(t)\int_t^{\min(s,t)} \frac{dF(u)}{(1-H(u)\ )R(u)}$$

where $1 - H(t) = Pr(\ T \le t \le X \mid X \le T)$ and $F(t) = Pr(\ X \le t, \delta = 1 \mid X \ge T\ )$.

The MRLF, $e(t)$ is defined to bet expected value remaining lifetime given survival rate of age t as follows:

$$e(t)\ =\ E\ (\ X - t \mid X > t\ )\ =\ (\ S(t))^{-1}\int_t^{\infty} S(u)du$$

Yang(1977) considered the truncated MRLF $e_t^T = \int_i^T S(s)\ ds/S(t)$, $H(T) < 1$ and studied the estimation procedure of it. Especially Yang(1978) showed the asymptotic properties of estimator, $\hat{e}_t^T$ using PL1 estimator on [0, T] in a random censoring model.

Kumazawa(1987) defined $\tau_F = \sup_t\{t \mid F(t) < 1\}$, $\hat{e}_t^{\tau_F} = \frac{\int_t^{\tau_F} S(s)ds}{S(t)}$ And he also proposed estimator $\hat{e}_t^{PL1}$ of $e_T^{\tau_F}$ using PL1 estimator with asymptoric properties in random censoring model on the whole line.

In this study, the estimator of MRLF, $\hat{e}_t^{PL2}$ of $e_T^{\tau_F}$ using PL2 estimator on LTRC is proposed and showed the asymptotic properties.

Let $T^* = \max_{1 \le i \le n} \{X_i\}$ and the proposed estimator $\hat{e}_t^{PL2}$ of $e_t^{\tau_F}$ using PL2 estimator is

$$\hat{e}_t^{PL2}\ =\ \frac{\int_t^{T^*} \hat{S}_{PL2}(s)\ ds}{\hat{S}_{PL2}(t)}$$

3. Asymptotic properties of the proposed estimator

Using Lemma 2.1. the consistency and asymptotic normality of $\hat{e}_t^{PL}(t)$ are proved.

Theorem 3.1 If

$$\sqrt{n}\int_{T^*}^{\tau_F} S(s)\ ds\ \rightarrow p\ 0 \quad \text{as} \quad n\ \rightarrow\ \infty$$

then

$$\sup_{0 \le t \le T^*} |\hat{e}_t^{PL2} - e_t^{\tau_F}| \to p\, 0 \quad \text{as} \quad n \to \infty$$

Proof. It is proved easily by lemma 2.1(a) and assumption $\sqrt{n} \int_{T^*}^{\tau_F} S(s)\, ds \to p\, 0$

Now, the weak convergency of $\hat{e}_t^{PL2}(t)$ of $e_T^{\tau_F}$ is proved. To show the weak convergency of $\hat{e}_t^{PL2}(t)$, $Z_n(t) \equiv \sqrt{n}\ (S(t) - \hat{S}_{PL2}(t)) \ / \ S(t)$ converges Gaussian process in lemma 2.1 and $S(t)Z_n(t)$ also converges Gaussian process with mean 0 and $Cov\ (Z^*(s),\ Z^*(t))$. For a given interval,[0, T], $H(T) < 1$, $Z \equiv B\ (V)$ converges to Brownian motion process with $V(t) = \int_0^t \frac{dF(s)}{(1-F(s))\ S(s)}$. Using this result and the theorem 2.2. of Gill(1983), the weak convergency is given theorem 3.2.

Theorem 3.2 Let $h(t) = \int_t^{\tau_F} S(s)\, ds$.
Suppose 1) $\sqrt{n}\, h(T^*) \to p\, 0$
and 2) $\int_0^{\tau_H} h^2(s)\, dV(s) < \infty$
Then, for $0 \le t \le T^*$

$$\sqrt{n}\ (\hat{e}_t^{PL2} - e_t^{\tau_F}) \to p\, B^*(t) \equiv -\frac{\int_t^{\tau_F} h(s)\, dZ(s)}{S(t)}, \quad n \to \infty$$

weakly in $D\ [0,\ \tau_F\ )$, where B^* is mean zero Gaussian process given by

$$Cov\ (B^*(s), B^*(t)) = (S(s)\ S(t))^{-1} \int_{s \wedge t}^{\tau_F} h^2(u)\, d\, V(u)$$

Proof. Using above assumptions 1) and 2), the condition of theorem 2.2 of Gill (1983) is proved. And also, by the theorem A.1.2 of Fleming and Harrington(1991),

$$\begin{aligned} \sqrt{n}\ (\hat{e}_t^{PL2} - e_t^{\tau_F}) &= (\hat{S}_{PL2}(t)\ \ S(t))^{-1} \{-S(t) \int_t^{T^*} h(s)\, dZ_n(s) \\ &\quad + S(t)\ Z_n(T^*)\ h(T^*) - S(t)\ \sqrt{n}\ h(T^*)\} \end{aligned}$$

is shown. Using the assumption 1), it is proved.

4. Simulation Study

The small sample performances of the proposed estimator $\hat{e}_t^{PL2}$ for MRLF is investigated in the aspects of the biases and mean squared error (MSE's) via Monte Carlo simulation.

The simulation is performed to compare the small sample properties of $\hat{e}_t^{PL2}(t)$ with those of the Kaplan-Meier type estimator $\hat{e}_t^{PL1}(t)$ based on the Kaplan-Meier estimator $\hat{S}_{PL1}(t)$ for the survival function $S(t)$ obtained by ignoring the truncation effects with Weibull distribution with constant failure rate, decreasing failure rate and increasing failure rate according to the scale and shape parameters. The censoring distributions are supposed to be exponential and uniform with the censoring rate about 10% and 30%. For sample size n=20, 50, the procedure is repeated 500 times to get estimates of biases and MSE's of the estimators $\hat{e}_t^{PL2}(t)$ and $\hat{e}_t^{PL1}(t)$ such that $t = S^{-1}(p)$, $p = 0.9, 0.8, \cdots, 0.1$.

The results are listed in Table 1 through Table 8. From these tables, the following facts are drawn.

(1) For the case of $Weib(1.0,\ 1.0)$ and the censoring distribution in exponential, the product-limit estimator $\hat{e}_t^{PL2}(t)$ tends to have smaller bias and MSE than the Kaplan-Meier type estimator $\hat{e}_t^{PL1}(t)$ except in the area of upper tail of the distribution with censoring rate 10% and $n = 20$. But, $\hat{e}_t^{PL}(t)$ is more efficient than $\hat{e}_t^{PL1}(t)$ ith the censoring rate 30% regardless of sample size. In the case of censoring distribution is given as uniform, $\hat{e}_t^{PL2}(t)$ always seems to be more efficient than $\hat{e}_t^{PL1}(t)$ with respect to both bias and MSE.

(2) For the case of $Weib(1.0,\ 1.5)$ and the censoring distribution is exponential or uniform with $n = 20$ and the censoring rate 10%, the product-limit type estimator $\hat{e}_t^{PL2}(t)$ has smaller bias than the Kaplan-Meier type estimator $\hat{e}_t^{PL1}(t)$ .But with respect to MSE, $\hat{e}_t^{PL1}(t)$ seems to be little more efficient than $\hat{e}_t^{PL2}(t)$ in the area of upper tail of the distirbution. In the case of $n = 50$, $\hat{e}_t^{PL2}(t)$ has tendency to be more efficient than $\hat{e}_t^{PL1}(t)$ regardless of the censoring rate.

Through the above results, it can be concluded that with data having the truncation effect, the product-limit type estimator seems to be more reasonable than Kaplan-Meier type estimator for the mean residual life function in the aspect of both biases and MSE's.

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Table 1: The Estimated Biases and MSE's (n=20)
lifetime : Weibull(1.0, 1.0) Truncation : Exp(2.0)
Censoring : Exp(.11) Censoring : Exp(.429) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	-.0691	-.0691	-.6354	-.6354
	MSE	.9906	.9906	.8316	.8316
0.2	Bias	.1367	.1367	-,1885	-.1885
	MSE	.5076	.5076	.3914	.3914
0.3	Bias	.1597	.1592	-.0209	-.0209
	MSE	.3628	.3608	.1975	.1975
0.4	Bias	.1268	.1259	.0065	.0060
	MSE	.2154	.2121	.1256	.1253
0.5	Bias	.1366	.1353	.0424	.0417
	MSE	.1668	.1629	.1097	.1093
0.6	Bias	.1517	.1493	.0703	.0687
	MSE	.1519	.1488	.1017	.1010
0.7	Bias	.1784	.1749	.1134	.1108
	MSE	.1367	.1336	.1077	.1070
0.8	Bias	.2211	.2169	.1636	.1604
	MSE	.1413	.1378	.1118	.1070
0.9	Bias	.2898	.2848	.2362	.2323
	MSE	.1755	.1710	.1439	.1420

Table 2: The Estimated Biases and MSE's (n=50)
lifetime : Weibull(1.0, 1.0) Truncation : Exp(2.0)
Censoring : Exp(.111) Censoring : Exp(.429) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	.1166	.1166	-.2169	-.0169
	MSE	.3944	.3944	.5446	.5446
0.2	Bias	.0630	.0626	-.0169	-.0169
	MSE	.1262	.1254	.2030	.2030
0.3	Bias	.0538	.0527	.0097	.0097
	MSE	.0852	.0837	.1570	.1570
0.4	Bias	.0688	.0672	.0207	.0203
	MSE	.0703	.0685	.1067	.1066
0.5	Bias	.0881	.0858	.0471	.0461
	MSE	.0614	.0596	.0718	.0715
0.6	Bias	.1084	.1052	.0688	.0670
	MSE	.0508	.0489	.0549	.0541
0.7	Bias	.1464	.1420	.1096	.1070
	MSE	.0574	.0550	.0629	.0618
0.8	Bias	.1942	.1884	.1553	.1518
	MSE	.0728	.0694	.0722	.0707
0.9	Bias	.2614	.2547	.2250	.2207
	MSE	.0132	.0114	.0987	.0962

Table 3: The Estimated Biases and MSE's (n=20)

lifetime : Weibull(1.0, 1.5) Truncation : Exp(1.67)

Censoring : Exp(.119) Censoring : Exp(.432) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	-.0912	-.0182	-.2133	-.2133
	MSE	.2303	.2303	.1902	.1902
0.2	Bias	.1350	.1350	.1055	.1055
	MSE	.1117	.1117	.1279	.1279
0.3	Bias	.1981	.1981	.1889	.1889
	MSE	.1007	.1007	.0929	.0929
0.4	Bias	.2356	.2354	.2090	.2090
	MSE	.1057	.1056	.0823	.0823
0.5	Bias	.2668	.2665	.2414	.2414
	MSE	.1082	.1080	.0866	.0866
0.6	Bias	.2895	.2892	.2758	.2757
	MSE	.1211	.1209	.1031	,1030
0.7	Bias	.3218	.3209	.3075	.3069
	MSE	.1397	.1390	.1206	.1201
0.8	Bias	.3524	.3509	.3439	.3430
	MSE	.1543	.1529	.1030	.1421
0.9	Bias	.3977	.3957	.3802	.3788
	MSE	.1881	.1861	.1694	.1682

Table 4: The Estimated Biases and MSE's (n=50)

lifetime : Weibull(1.0, 1.5) Truncation : Exp(1.67)

Censoring : Exp(.119) Censoring : Exp(.432) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	.0351	.0351	.0045	.0045
	MSE	.0668	.0668	.0820	.0820
0.2	Bias	.1093	.1093	.1223	.1223
	MSE	.0404	.0404	.0533	.0533
0.3	Bias	.1585	.1584	.1712	.1712
	MSE	.0458	.0458	.0550	.0550
0.4	Bias	.2014	.2012	.1991	.1991
	MSE	.0567	.0566	.0588	.0588
0.5	Bias	.2368	.2364	.2321	.2321
	MSE	.0684	.0682	.0676	.0676
0.6	Bias	.2667	.2662	.2587	.2585
	MSE	.0823	.0819	.0767	.0765
0.7	Bias	.3000	.2989	.2911	.2906
	MSE	.1008	.1001	.0942	.0939
0.8	Bias	.3341	.3325	.3202	.3193
	MSE	.1215	.1204	.1129	.1122
0.9	Bias	.3774	.3751	.3631	.3618
	MSE	.1518	.1500	.1424	.1414

Table 5: The Estimated Biases and MSE's (n=20)
lifetime : Weibull(1.0, 1.0) Truncation : Exp(2.0)
Censoring : Uni(10.0) Censoring : Uni(3.197) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	-.1350	-.1350	-.8465	-.8465
	MSE	.6934	.6934	.7815	.7815
0.2	Bias	.1225	.1225	-.3458	-.3458
	MSE	.3564	.3564	.2743	.2743
0.3	Bias	.0791	.0787	-.1320	-.1320
	MSE	.2048	.2041	.1427	.1427
0.4	Bias	.0811	.0799	-.0728	-.0728
	MSE	.1439	.1430	.1086	.1086
0.5	Bias	.0976	.0959	-.0318	-.0319
	MSE	.1070	.1059	.0798	.0799
0.6	Bias	.1211	.1183	.0125	.0121
	MSE	.1040	.1028	.0725	.0725
0.7	Bias	.1550	.1509	.0523	.0510
	MSE	.1020	.0999	.0622	.0622
0.8	Bias	.2046	.1991	.1090	.1070
	MSE	.1152	.1121	.0665	.0663
0.9	Bias	.2765	.2704	.1739	.1715
	MSE	.0345	.0330	.0202	.0202

Table 6: The Estimated Biases and MSE's (n=50)
lifetime : Weibull(1.0, 1.0) Truncation : Exp(2.0)
Censoring : Uni(10.0) Censoring : Uni(3.197) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	.0406	.0406	-.6583	-.6583
	MSE	.2120	.2120	.5133	.5133
0.2	Bias	.0412	.0406	-.2271	-.2271
	MSE	.0954	.0952	.1016	.1016
0.3	Bias	.0358	.0348	-.1389	-.1389
	MSE	.0644	.0639	.0646	.0646
0.4	Bias	.0610	.0592	-.0810	-.0813
	MSE	.0514	.0508	.0433	.0433
0.5	Bias	.0746	.0719	-.0386	-.0392
	MSE	.0453	.0446	.0357	.0358
0.6	Bias	.1040	.1004	-.0036	-.0024
	MSE	.0448	.0438	.0298	.0298
0.7	Bias	.1404	.1357	.0468	.0449
	MSE	.0490	.0473	.0263	.0261
0.8	Bias	.1886	.1825	.0979	.0951
	MSE	.0630	.0630	.0331	.0326
0.9	Bias	.2570	.2499	.1674	.1640
	MSE	.0086	.0080	.0496	.0484

Table 7: The Estimated Biases and MSE's (n=20)
lifetime : Weibull(1.0, 1.5) Truncation : Exp(1.67)
Censoring : Uni(9.024) Censoring : Uni(3.002) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	-.0742	-.0742	-.3012	-.3012
	MSE	.2004	.2004	.1567	.1567
0.2	Bias	.1643	.1643	.0323	.0323
	MSE	.1015	.1015	.0756	.0756
0.3	Bias	.1956	.1954	.1451	.1451
	MSE	.0869	.0866	.0596	.0596
0.4	Bias	.2339	.2335	.1924	.1924
	MSE	.0915	.0911	.0761	.0761
0.5	Bias	.2627	.2623	.2281	.2280
	MSE	.0985	.0980	.0868	.0867
0.6	Bias	.2885	.2878	.2675	.2671
	MSE	.1121	.1114	.1019	.1015
0.7	Bias	.3159	.3148	.2934	.2929
	MSE	.1274	.1264	.1125	.1120
0.8	Bias	.3520	.3505	.3317	.3306
	MSE	.1503	.1488	.1356	.1349
0.9	Bias	.3960	.3941	.3776	.3764
	MSE	.1843	.1822	.1690	.1680

Table 8: The Estimated Biases and MSE's (n=50)
lifetime : Weibull(1.0, 1.5) Truncation : Exp(1.67)
Censoring : Uni(9.024) Censoring : Uni(3.002) Censoring rate : 10% Censoring rate : 30%

S(t)		K-M	P-L	K-M	P-L
0.1	Bias	.0168	.0168	-.1385	-.1385
	MSE	.0443	.0443	.0728	.0728
0.2	Bias	.1154	.1154	.0673	.0673
	MSE	.0348	.0348	.0255	.0255
0.3	Bias	.1556	.1555	.1305	.1305
	MSE	.0396	.0395	.0333	.0333
0.4	Bias	.2049	.2046	.3745	.1743
	MSE	.0547	.0545	.0446	.0446
0.5	Bias	.2374	.2369	.2125	.2122
	MSE	.0685	.0682	.0580	.0579
0.6	Bias	.2700	.2693	.2489	.2484
	MSE	.0837	.0832	.0720	.0718
0.7	Bias	.2998	.2985	.2777	.2770
	MSE	.0995	.0986	.0855	.0850
0.8	Bias	.3378	.3360	.3138	.3127
	MSE	.1234	.1221	.1070	.1063
0.9	Bias	.3795	.3772	.3564	.3549
	MSE	.1530	.1512	.1358	.1347