Journal of the Korean
Data & Information Science Society
1999, Vol. 10, No. 1, pp. 19 ~ 27

Asymptotic Distribution of the Estimator for Rotation in Orientation-Shift Model on the Sphere [1]

Yang-Kyu Shin [2]

Abstract

An orientation-shift model for unit random vectors in R^3 is defined. The Euler angles are introduced to parametrize orientation-shifts and their moment estimators are found. Through the analytic perturbation method, the asymptotic distributions of the estimators are obtained.

Key Words and Phrases: Orientation-shift model, Rotation, Euler-angles, Moment estimator, Asymptotic distribution.

1. Introduction

In analyzing the spherical data, it is important to formulate a location model on the sphere in R^3. Let Y be a spherical random vector with a probability density function g. By employing an analogy between the random vector Y with probability density function g and the unit spherical shell with the same mass density function, an orientation-shift from Y to ΓY by a proper rotation $\Gamma \in O^+(3)$, where $O^+(3)$ is the group of proper rotations, or the group of 3×3 orthogonal matrices with determinant 1, can be viewed as the equivalent rotation of the spherical shell(Kim(1972), Watson(1968)). It is customary to use an orthogonal set of eigenvectors

[1]This research was supported in part by Korea Institute of Oriental Medicine Grant for 1998
[2]Associate Professor, Dept. of Information Stitistics, Kyungsan University, Kyungpook, 712-240, Korea

of the inertia matrix of a rigid body as a body-coordinate system to describe such rotation(Tyler(1981), Watson(1983)).

When independent observations are made on ΓY, the sample eigenvectors are used to obtain an estimate $\widehat{\Gamma}_n$ of the unknown shift Γ. As in the study of rigid bodies, Euler angles Θ parametrize proper rotation. And the analytic perturbation method supply the non-rigorous nature of the δ-method which is commonly used to treat perturbation of eigenvectors.(Kato(1966)). This paper is an attempt to formulate a location-model on the unit sphere in R^3 based on the Euler angles. In section 2, orientation-shift model on the sphere is discussed. In section 3, a moment estimate $\widehat{\Theta}_n$ of the Euler angles is obtained from a moment estimator of $\widehat{\Gamma}_n$. In the final section, through the analytic perturbation method, asymptotic distributions of the Euler angles are obtained.

2. Orientiation-Shift Model

Let λ be the invariant measure on $S^2 = \{\| x \| = 1, x \in R^3\}$, the unit sphere in $R^3 : d\lambda(\alpha, \beta) = sin\phi d\alpha d\beta$, where (α, β) are the polar angles of $x = (x_1, x_2, x_3) \in S^2$ so that $x_1 = sin\alpha cos\beta, x_2 = sin\alpha sin\beta$, and $x_3 = cos\alpha$. From now on, whenever a unit random vector on S^2 is mentioned, the vector will be assumed to have a probability density function with respect to λ. Suppose a sample is taken on a unit random vector X with probability density function of the form

$$f(x) = g(\Gamma^{-1}x), x \in S^2, \Gamma \in O^+(3),$$

where the orientation-shift Γ from g to f is to be estimated from the sample. Let Y be a unit random vector with probability density function g. Define

$$Y_R = \{\Gamma Y : \Gamma \in O^+(3)\}.$$

Similarly, noting that the probability density function of ΓY is $y \circ \Gamma^{-1}$, define $g_R = \{g \circ \Gamma^{-1} : \Gamma \in O^+(3)\}$. Then Y_R is the orientation family generated by Y, and g_R is the orientation family generated by g. Not all the members of g_R are distinct in the presence of an underlying density, inducing natural equivalence relations on $O^+(3)$ and in Y_R :

$$\Gamma_1 \sim \Gamma_2 \quad if \;\; and \;\; only \;\; if \quad g \circ \Gamma_1^{-1} = g \circ \Gamma_2^{-1};$$
$$\Gamma_1 Y \sim \Gamma_2 Y \quad if \;\; and \;\; only \;\; if \quad \Gamma_1 = \Gamma_2.$$

Then a proper rotation Γ is the orientation-shift from Y (or g) to ΓY (or $g \circ \Gamma^{-1}$). This leads to the following description : an orientation-shift model consists of a reference vector Y, the group of orientation-shifts $O^+(3)$, the orientation family Y_R generated by Y through the orientation-shifts in $O^+(3)$, and the equivalence classes in Y_R induced by the underlying symmetry. In considering a sample from the common density $g \circ \Gamma^{-1}$ with Γ to be estimated, it is now clear that $\bar{\Gamma}$ rather than the particular member Γ is what needs to be estimated; it is enough to find an estimate for any $\Gamma_1 \sim \Gamma$. It should be noted the equivalence relations depend on the choice of a reference vector Y which could have been any other member in Y_R , for we clearly have for any $\Gamma \in O^+(3)$. Since changing the reference vectors from Y to ΓY amounts to making the coordinate transformation $\Gamma^{-1} = \Gamma'$, the equivalence classes in Y_R (or in $O^+(3)$) undergo a similarity transformation under such change. Though different reference vectors give rise to different equivalence structures in Y_R as we saw, it is largely up to us to decide on a canonical choice of a reference vector Y which would provide a simplest equivalence structure, for we can always go from one structure to another by a simple transformation.

To discuss how to recover $\bar{\Gamma}$, we first need some notations. Define

$$\mu_{\Gamma Y} = E(\Gamma Y) \text{ and } M_{\Gamma Y} = E(\Gamma Y)(\Gamma Y)'. \tag{1}$$

Let

$$\Delta = diag(d_1, d_2, d_3), \tag{2}$$

where $d_1 \leq d_2 \leq d_3$ are the observed eigenvalues of $M_{\Gamma Y}$, for any $\Gamma \in O^+(3)$. Depending on the values of (d_1, d_2, d_3) in (2), we have different types of spherical distributions. In this paper, we will restrict our attention to $d_1 < d_2 < d_3$. Define, using (1) and (2),

$$O_{\Gamma Y} = \{\Sigma_{\Gamma Y} \in O^+(3) : M_{\Gamma Y} = \Sigma_{\Gamma Y} \Delta \Sigma_Y', \Sigma_{\Gamma Y} \Sigma_{\Gamma Y}' = I\},$$

where $\Sigma_{\Gamma Y}$ is an orthogonal matrix whose columns are the oriented, ordered and orthonormalized eigenvectors of $M_{\Gamma Y}$. We may write, for convenience, μ_Γ, M_Γ, O_Γ and Σ_Γ, for $\mu_{\Gamma Y}$, $M_{\Gamma Y}$, $O_{\Gamma Y}$ and $\Sigma_{\Gamma Y}$, respectively. Let C_S be the class of spherical distributions satisfying the condition: whenever $\mu_Y = \Gamma\mu_Y$ and $O_\Gamma = \Gamma O_Y, Y \sim \Gamma Y$.

We are now in a position to discuss how to recover $\bar{\Gamma}$ given the values (μ_Y, O_Y) and (μ_Γ, O_Γ) , for $Y \in C_S$. First we need to find a pair $(\Sigma_Y, \Sigma_\Gamma) \in O_Y \times O_\Gamma$ such that $\Gamma \sim \Sigma_\Gamma \Sigma_Y'$. Such a pair can be found by imposing the following invariance condition on the measure:

I. $\mu_Y \cdot \sigma_{jY} = \mu_\Gamma \cdot \sigma_{j\Gamma}$ for all j.

A pair $(\Sigma_Y, \Sigma_\Gamma)$ in satisfying I will be called a "compatible" pair. For the compatible pairs, we have the following theorem 1 (Kim (1978)).

Theorem 1. Suppose $Y \in C_S$ and let $(\Sigma_Y, \Sigma_\Gamma) \in O_Y \times O_\Gamma$ be a compatible pair. Then $\Gamma \sim \Sigma_\Gamma \Sigma_Y'$ allowing us to represent $\bar{\Gamma}$ by $\Sigma_\Gamma \Sigma_Y'$.

Theorem 1 suggests that, by choosing Y so as the identity matrix I to belong to O_Y and then by selecting Σ_Γ from O_Y and then by selecting Σ_Γ from O_Y so as to form a compatible pair with I, $\bar{\Gamma}$ can be recovered simply by Σ_Γ. Therefore, with canonically selected Y, Σ_Y and $\Sigma_{\Gamma Y}$, the shift $\bar{\Gamma}$ from Y to ΓY is recovered simply by

$$\Gamma_1 = \Sigma_{\Gamma Y}. \tag{3}$$

3. Estimation

Let $X_1, X_2, \cdots, X_n$ be the independent observations on $X = \Gamma Y$ for some unknown $\Gamma \in O^+(3)$, where $Y \in C_S$ and is canonically selected. The shift $\bar{\Gamma}$ can be recovered from (μ_Y, O_Y) and (μ_Γ, O_Γ) by (1). Heuristically at least, once an estimate $(\widehat{\mu}_n, \widehat{O}_n)$ of (μ_Γ, O_Γ) is found, we would have an estimate $\widehat{\Gamma}_n$ of $\bar{\Gamma}$ simply by substituting $(\widehat{\mu}_n, \widehat{O}_n)$ for (μ_Γ, O_Γ) in (3). For $X_1, X_2, \cdots, X_n$, we can define the sample counterparts of μ_Γ and O_Γ :

$$\widehat{\mu}_n = \frac{1}{n}\sum_{j=1}^{n} X_j \tag{4}$$

and

$$\widehat{O}_n = \{\widehat{\Sigma}_n \in O^+(3); \widehat{M}_n = \widehat{\Sigma}_n \widehat{\Delta}_n \widehat{\Sigma}_n', \widehat{\Sigma}_n \widehat{\Sigma}_n' = I\}, \tag{5}$$

where $\widehat{M}_n = \frac{1}{n}\sum_{j=1}^{n} X_j X_j'$ and $\widehat{\Delta}_n = diag(\widehat{d}_{1n}, \widehat{d}_{2n}, \widehat{d}_{3n})$ with the strictly ordered (w.p. 1) eigenvalues $\widehat{d}_{1n} < \widehat{d}_{2n} < \widehat{d}_{3n}$ of $\widehat{M}_n$. And the sample invariance condition is $\widehat{I}_n$. $(\mu_Y \cdot \sigma_{jY})(\widehat{\mu}_n \cdot \widehat{\sigma}_{jn}) \geq 0$ for all j . Accordingly, by (3), we take the canonically selected

$$\widehat{\Gamma}_n = \widehat{\Sigma}_n \tag{6}$$

as our estimate of $\bar{\Gamma}$.

3.1 Parametrization by Euler Angles

Since the columns of orthogonal matrices satisfy three orthogonality conditions between themselves and three more normality constraints on each column, $O^+(3)$ can be parametrized by a suitable set of three independent parameters. The usefulness of the set of Euler angles in describing rotational motions of rigid bodies is proved in classical mechanics. And Oliner, Rudebusch and Sichel(1996) asses the stability of empirical Euler equations for investment. Any proper rotation Γ can be performed as a succession of three elementary rotations about the coordinate vectors by Euler angles. They consist of : the first rotation which is made about the e_3-axis by an angle ϕ, described by $R_3(\phi)$; the second rotation about the e_2-axis by an angle θ, described by $R_2(\theta)$; the third and final rotation about the e_3-axis by an angle ψ, described by $R_3(\psi)$. The complete rotation is given by $\Gamma(\Theta) = R_3(\psi)R_2(\theta)R_3(\phi)$ with $\Theta = (\phi, \theta, \psi)'$:

$$\begin{aligned}
\Gamma(\Theta) &= R_3(\psi)R_2(\theta)R_3(\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi & \sin\theta\cos\psi \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & -\sin\phi\cos\theta\cos\psi + \cos\phi\cos\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}$$

3.2 Moment Estimators of Θ

When parametrized, the recovery (3) read, putting $\Gamma_1 = \Gamma(\Theta_1)$ with $\Theta_1 = (\phi_1, \theta_1, \psi_1)'$,

$$R_3(\psi_1)R_2(\theta_1)R_3(\phi_1) = \Sigma_\Gamma. \tag{7}$$

On the other hand, (6) yield, upon setting $\Gamma(\widehat{\Theta}_n) = \widehat{\Gamma}_n$ with $\widehat{\Theta}_n = (\widehat{\phi}_n,\ \widehat{\theta}_n, \widehat{\psi}_n)'$,

$$R_3(\widehat{\psi}_n)R_2(\widehat{\theta}_n)R_3(\widehat{\phi}_n) = \widehat{\Sigma}_n. \tag{8}$$

Then, (8) gives an estimate $\widehat{\Theta}_n$ of the Euler angles Θ_1 of Γ_1 (or of $\bar{\Gamma}$). Let us solve (8). Postmultiplying (8) by e_3, we have

$$R_3(\widehat{\psi}_n)R_2(\widehat{\theta}_n)e_3 = \widehat{\sigma}_{3n}. \tag{9}$$

On the other hand, postmultiplying the transpose of (8) by e_3, we have

$$R_3(-\widehat{\psi}_n)R_2(-\widehat{\theta}_n)e_3 = \widehat{\sigma}^{3n}, \tag{10}$$

where $\widehat{\sigma}^{3n}$ is the vector of the third row of $\widehat{\Sigma}_n$. The equations (9) and (10) show that $(\widehat{\theta}_n, \widehat{\psi}_n)$ are simply the polar angles of $\widehat{\sigma}_{3n}$ and $(-\widehat{\theta}_n, -\widehat{\psi}_n)$ those of $\widehat{\sigma}^{3n}$. Hence $\widehat{\Theta}_n$ is explicitly given by

$$\begin{aligned}
\widehat{\psi}_n &= \tan^{-1}[\ -\frac{\widehat{\sigma}_{32,n}}{\widehat{\sigma}_{31,n}}\ ], \\
\widehat{\theta}_n &= \tan^{-1}[\ \frac{\sqrt{1-\widehat{\sigma}_{33,n}}}{\widehat{\sigma}_{33,n}}\ ], \\
\widehat{\phi}_n &= \tan^{-1}[\ -\frac{\widehat{\sigma}_{23,n}}{\widehat{\sigma}_{13,n}}\ ],
\end{aligned}$$

where $\widehat{\Sigma}_n = (\widehat{\sigma}_{ij,n})$ is canonically chosen.

4. Asymptotic Distribution of Θ

Let us find the asymptotic distribution of the Euler angles Θ for $d_1 < d_2 < d_3$. Let $\Sigma_\Gamma, \widehat{\Sigma}_n, \Theta_1$ and $\widehat{\Theta}_n$ be as defined in (7) and (8). Put $\delta_n\sigma_{ij} = \sqrt{n}(\widehat{\sigma}_{ij,n} - \sigma_{ij,\Gamma})$ and $\delta_n\Theta = \sqrt{n}(\widehat{\Theta}_n - \Theta_1)$.

Theorem 2. Assume that $\theta_1 \neq 0$ or π. Then

$$\delta_n\Theta \xrightarrow{\mathcal{L}_n} -Ws_D \sim N(0, WC[s_D]W'),$$

where $s_D = (d_{23}s_{23}, d_{31}s_{31}, d_{12}s_{12})'$ with $d_{jk} = (d_j - d_k)^{-1}$ and $W = V^{-1}$. The covariance structure $C[s_D]$ is given by

$cov(d_{ij}s_{ij}, d_{kl}s_{kl}) = E(Y_iY_jY_kY_l) - E(Y_iY_j)E(Y_kY_l), 1 \leq i,j,k,l \leq 3$. And

$$V = \begin{bmatrix} 0 & \sin\phi & -\cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix}.$$

Proof. Since $\Gamma(\Theta_1) = \Sigma_\Gamma$ by (7), Taylor expansion yields

$$\delta_n\sigma_{ij} + o_p(1) = \delta_n(\Theta) \cdot \frac{d}{d\Theta_1} r_{ij}(\Theta_1) = \delta_n(\Theta) \cdot \frac{d}{d\Theta_1}(e_j{}'\Gamma(\Theta_1)'e_i), \tag{11}$$

where $\Gamma(\Theta_1) = r_{ij}(\Theta_1)$. Since $\frac{d}{d\theta}h(\Gamma(\Theta)^{-1}x) = V'(y \times \frac{d}{dy}h(y))$ in $h \in C'(s^2)$, for $h(\Gamma(\Theta)^{-1}e_i) = e_j'\Gamma(\Theta)^{-1}e_i$, we have

$$\begin{aligned} \frac{d}{d\theta_1}(e_j'\Gamma(\theta_1)'e_i) &= V'(\sigma^{i\Gamma} \times \frac{d}{d\sigma^{i\Gamma}} e_j{}'e^{i\Gamma}) \\ &= V'(e^{i\Gamma} \times e_j), \end{aligned} \tag{12}$$

where $\Sigma_\Gamma{}' = (\sigma^{1\Gamma}, \sigma^{2\Gamma}, \sigma^{3\Gamma})$.

On the other hand, since $\widehat{\Gamma}_n = \widehat{\Sigma}_n$, there corresponds a canonical choice Σ_Γ such that for each j,

$$\sqrt{n}(\widehat{\sigma}_{jn} - \sigma_{j\Gamma}) = \sum_{k(\neq j)} [d_{jk}\sigma_{j\Gamma}{}'\sqrt{n}(\widehat{M}_n - M_\Gamma)\sigma_{k\Gamma}]\sigma_{k\Gamma} + o_p(1) \tag{13}$$

with $d_{jk} = (d_j - d_k)^{-1}$. Since $\Sigma_\Gamma{}'X \sim \Gamma'X = Y$ for a canonical choice Σ_Γ, $\sqrt{n}(\widehat{M}_n - M_\Gamma)$ transforms under a coordinate transformations $\Sigma_\Gamma{}'$ to

$$s_{D,n} = \Sigma_\Gamma{}' \sqrt{n}(\widehat{M}_n - M_\Gamma)\Sigma_\Gamma \longrightarrow s \sim N(0, C[s_D]), \tag{14}$$

where $C[s_D]$ is given by $cov(d_{ij}s_{ij}, d_{kl}s_{kl}) = E(Y_iY_jY_kY_l) = E(Y_iY_j)E(Y_kY_l), 1 \leq i,j,k,l \leq 3$. By (13) and (14),

$$\begin{aligned}
\delta_n\sigma_{ij} + o_p(1) &= \sum_{k(\neq j)} [d_{jk}\sigma_{j\Gamma}{}' \sqrt{n}(\widehat{M}_n - M_\Gamma)\sigma_{k\Gamma}]\sigma_{ik,\Gamma} \\
&= \sum_{k(\neq j)} [d_{jk}\sigma_{j\Gamma}{}' \Sigma_\Gamma s_n \Sigma_\Gamma{}' \sigma_{k\Gamma}]\sigma_{ik,\Gamma} \\
&= \sum_{k(\neq j)} [d_{jk} e_j' s_n e_k]\sigma_{ik,\Gamma} \\
&= \sum_{k(\neq j)} d_{jk} s_{jk,\Gamma}\sigma_{ik,\Gamma} \\
&= (\sigma^{i\Gamma} \times s_{D,n}) \cdot e_j \\
&= -s_{D,n} \cdot (\sigma^{i\Gamma} \times e_j),
\end{aligned} \tag{15}$$

where $s_{D,n} = (d_{23}s_{23,n}, d_{31}s_{31,n}, d_{12}s_{12,n})'$. Now substituting (12) and (15) into (11) and comparing both sides, we have

$$-s_{D,n} = V\delta_n\Theta + o_p(1), \tag{16}$$

for $\{\sigma^{i\Gamma} \times e_j : 1 \leq i,j \leq 3\}$ clearly spans R^3. The equation (16) now yields, since V^{-1} exists by the assumptions on θ_1 ,

$$\delta_n\Theta = -V^{-1}s_{D,n} + o_p(1) \xrightarrow{\mathcal{L}_n} -V^{-1}s_D = -Ws_D \sim N(0, WC[s_D]W').$$

Reference

1. Kato, T. (1966). *Pertubation Theory for Linear Operators,(2nd ed.)*. Springer-Verlag, New York.

2. Kim, K. M. (1978). *Orientation-shift Model on the Sphere*. Ph.D. thesis. Univ. of Califorina, Berkely.

3. Oliner, S. D., Rudebusch, G. D. and Sichel, D. (1996). The Lucas critique revisited: Assessing the stability of empirical euler equations for investment, *Journal of Econometrics*, 70, 291-316.

4. Tyler, D. E. (1981). On the asymptotic inference for eigenvectors, *The Annals of Statistics*, 9, 725-736.

5. Watson, G. S. (1968). *Orientation Statistics in the Earth Science.* Technical Report No. 93. Univ. of Johns Hopkins, Maryland.

6. Watson, G. S. (1983). *Statistics on Spheres.* John Wiley and Sons, New York.