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On UMVU Estimator of Parameters in Lognormal Distribution

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Abstract

To estimate the mean and the variance of a lognormal distribution, Finney (1941) derived the uniformly minimun variance unbiased estimators(UMVUE) in the form of infinite series. However, the conditions $\sigma^2 > n$ and $\sigma^2 < \frac{n}{4}$ for computing $E(\hat{\theta}_{AM})$ and $E(\hat{\eta}^2_{AM})$ are necessary. In this paper, we give an alternative derivation of the UMVUE's.

Key Words and Phrases: Lognormal distribution, UMVU estimator.

1. Introduction

A demand on experimental fields is to estimate the effects of experimental treatments expressed in the original units. Especially, several authors dealt with this problem in the case such that the transformed variable follows a normal distribution with parameters μ and σ^2.

Let $X_1, X_2, \cdots, X_n$ be independently and identically distributed(i.i.d.) random variables having a lognormal distribution with mean θ and variance η^2. Finney(1941) obtained the uniformly minimum variance unbiased(UMVU) estimator of θ and η^2. Moreover, he obtained the expressions in powers to n^{-2} and n^{-1} of the variance of the UMVU estimators of θ and variance η^2, respectively. Under the same conditions Bradu and Mundlak(1970) obtained the UMVU estimator of the parametric functions of the following type $f_{\alpha,\beta}(\theta, \eta) = \exp(\alpha\theta + \beta\eta^2)$ $(\alpha, \beta :$ real numbers),

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which contain the mean, the median, the mode and the coefficient of variation. Crow(1977) obtained the UMVU estimator of the ratio of means of two lognormal variables, Shimizu and Iwosl(1981) showed that the UMVU estimators and their variances ¿from independent samples of lognormal distributions are concisely expressed using the hypergeometric functions. We study an alternative derivation of Finney's(1941) UMVU estimators of θ and η^2.

In this paper, we consider an alternative derivation of Finney's(1941) UMVU estimators of θ and η^2.

2. Priliminary and Lemma

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables having a lognormal distribution with mean θ and variance η^2, both being unknown. Let $Y_i = \ln X_i (i = 1, \cdots, n)$. Then Y_i is distributed as $N(\mu, \sigma^2)$ with

$$\begin{aligned} \theta &= \exp(\mu + \frac{\sigma^2}{2}) \\ \eta^2 &= \exp(2\mu + \sigma^2)\{\exp(\sigma^2) - 1\}. \end{aligned}$$

We know that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $S_Y^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$ are joint complete sufficient statistics for μ and σ^2. Moreover, we can obtain easily obtain the maximum likelihood estimators(MLE) of θ and η^2 as

$$\begin{aligned} \widehat{\theta}_M &= \exp(\bar{Y} + \frac{1}{2n}S_Y^2), \\ \widehat{\eta}_M^2 &= \exp(2\bar{Y} + \frac{1}{n}S_Y^2)\{\exp(\frac{1}{n}S_Y^2) - 1\}, \end{aligned}$$

respectively.

Since

$$
\begin{aligned}
E(\widehat{\theta}_M) &= \theta \exp\{-\frac{n-1}{n}\frac{\sigma^2}{2}\}(1-\frac{\sigma^2}{n})^{-\frac{n-1}{2}} \\
E(\widehat{\eta}^2_M) &= \eta^2 \exp\{(\frac{2}{n}-1)\sigma^2\}\{\exp(\sigma^2)-1\}^{-1} \\
& \quad \times\{(1-\frac{4\sigma^2}{n})^{-\frac{n-1}{2}} - (1-\frac{2\sigma^2}{n})^{-\frac{n-1}{2}}\},
\end{aligned}
$$

$\widehat{\theta}_M$ and $\widehat{\eta}^2_M$ are biased.

On the other hand, Finney(1941) obtained the UMVU estimators $\widehat{\theta}_{AM}$ and $\widehat{\eta}^2_{AM}$ of θ and η^2 respectively.
That is,

$$
\begin{aligned}
\widehat{\theta}_{AM} &= \exp(\bar{Y}) f(\frac{1}{2n}S^2_Y) && (1) \\
\widehat{\eta}^2_{AM} &= \exp(2\bar{Y})\Big\{f(\frac{2}{n}S^2_Y) - f(\frac{n-2}{n(n-1)}S^2_Y)\Big\}, && (2)
\end{aligned}
$$

where

$$
f(t) = 1 + t + \tfrac{n-1}{n+1}\tfrac{t^2}{2!} + \tfrac{(n-1)^2}{(n+1)(n+3)}\tfrac{t^3}{3!} + \cdots. \qquad (3)
$$

Moreover, he obtained their asymptotic variances :

$$
\begin{aligned}
Var(\widehat{\theta}_{AM}) &\sim \frac{1}{n}(\sigma^2 + \frac{1}{2}\sigma^4)\exp(2\mu+\sigma^2) \\
Var(\widehat{\eta}^2_{AM}) &\sim \frac{2\sigma^2}{n}\exp(4\mu+2\sigma^2)\{2[\exp(\sigma^2)-1]^2 \\
& \quad + \sigma^2[2\exp(\sigma^2)-1]^2\}.
\end{aligned}
$$

Remark The conditions $\sigma^2 < n$ and $\sigma^2 < \frac{n}{4}$ for computing $E(\hat{\theta}_{AM})$ and $E(\hat{\eta}^2_{AM})$ are necessary in Finney(1941) and Kendall and Stuart(1979).

Now we consider the Helmert orthogonal transformation. Assume, without loss of generality, $Y_1, Y_2, \cdots, Y_n$ are distributed as $N(0,1)$. Let $Y' = (Y_1, Y_2, \cdots, Y_n)$ and $Z' = (Z_1, Z_2, \cdots, Z_n)$ be $n \times 1$ random vectors such that $Z = \Gamma Y$, where

$$\Gamma = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{-1\cdot 2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots \frac{-(n-1)}{\sqrt{(n-1)n}} & \end{bmatrix}$$

is $n \times n$ Helmert orthogonal matrix. Then we know that $Z'Z = Y'Y$, and $Z_1, Z_2, \cdots, Z_n$ are i.i.d. $N(0,1)$.

Lemma Let X be a random variable that is $N(0,1)$ and Y be a random variable that has chi-squared distribution with $k(\geq 1)$ degree of freedom. And assume that X and Y are independent. Let $V = X/\sqrt{X^2+Y}$. Then the p.d.f. of V is

$$f(v) = \left[B(\frac{1}{2}, \frac{k}{2})\right]^{-1}(1-v^2)^{\frac{k-2}{2}}, \qquad -1 < v < 1,$$

and $E(V^{2m+1}) = 0$ for $m = 0, 1, 2, \cdots$, where $B(\frac{1}{2}, \frac{k}{2})$ is Beta function.

Proof. The joint p.d.f. of X and Y is

$$\begin{aligned} f(x,y) &= \frac{1}{\sqrt{\pi}\Gamma(\frac{k}{2})2^{\frac{k+1}{2}}}\exp\Big\{-\frac{x^2}{2}\Big\}y^{\frac{k-2}{2}}\exp\Big\{-\frac{y}{2}\Big\}, \\ &- \infty < x < \infty, \quad 0 < y < \infty. \end{aligned}$$

Let $v = x/\sqrt{x^2+y}$ and $w = y$. Then the Jocobian of the transformation is $w^{\frac{1}{2}}(1-v^2)^{-\frac{3}{2}}$. Hence, the joint p.d.f. of V and W is

$$\begin{aligned} f(v,w) &= \frac{1}{\sqrt{\pi}\Gamma(\frac{k}{2})2^{\frac{k+1}{2}}}\exp\Big\{-\frac{w}{2(1-v^2)}\Big\}w^{\frac{k-1}{2}}(1-v^2)^{-\frac{3}{2}}, \\ & -1 < v < 1, \qquad 0 < w < \infty. \end{aligned}$$

Therefore, the marginal p.d.f. of V is

$$\begin{aligned} f(v) &= \frac{1}{\sqrt{\pi}\Gamma(\frac{k}{2})2^{\frac{k+1}{2}}}(1-v^2)^{-\frac{3}{2}}\int_0^{\infty}\exp\Big\{-\frac{w}{2(1-v^2)}\Big\}w^{\frac{k-1}{2}}dw \\ &= \frac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\Gamma(\frac{k}{2})}(1-v^2)^{\frac{k-2}{2}}, \qquad -1 < v < 1. \end{aligned}$$

Furthermore

$$\begin{aligned} E(V^{2m+1}) &= \frac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\Gamma(\frac{k}{2})} \int_{-1}^{1} v^{2m+1}(1-v^2)^{\frac{k-2}{2}} dv \\ &= 0 \qquad \text{for} \qquad m = 0, 1, 2, \cdots, \end{aligned}$$

because the integrand is an odd function. ∎

3. UMVU Estimators of θ and η^2

First, we consider the UMVU estimator of θ. We can show easily that $\exp(Y_n)$ is an unbiased estimator of θ. Hence, by the Rao-Blackwell theorem, $E[\exp(Y_n) \mid \bar{Y}, S_Y^2]$ is the UMVU estimator of θ. By the Helmert orthogonal transformation $Z = \Gamma Y$, $Z_1^2 = n(\bar{Y})^2$, $S_Y^2 = \sum_{i=2}^{n} Z_i^2$ and $Y_n - \bar{Y} = -\sqrt{\frac{n-1}{n}} Z_n$. Therefore,

$$\frac{Y_n - \bar{Y}}{S_Y} = -\frac{\sqrt{\frac{n-1}{n}} Z_n}{(\sum_{i=2}^{n} Z_i^2)^{\frac{1}{2}}} = -\sqrt{\frac{n-1}{n}} U,$$

where $U = \frac{Z_n}{(\sum_{i=2}^{n} Z_i^2)^{\frac{1}{2}}} = \frac{Z_n}{(Z_n^2 + \sum_{i=2}^{n-1} Z_i^2)^{\frac{1}{2}}}$.

Hence, by Lemma, the p.d.f. of U is

$$f(u) = \Big[B(\frac{1}{2}, \frac{n-2}{2})\Big]^{-1} (1-u^2)^{\frac{n-4}{2}}, \qquad -1 < u < 1.$$

Furthermore, by Basu's Theorem(Lehmann(1983)), $\frac{(Y_n - \bar{Y})}{S_Y}$ is independent of $\bar{Y}$ and S_Y^2.

Now we consider $E[\exp(Y_n) \mid \bar{Y}, S_Y^2]$. Then

$$\begin{aligned} &E[\exp(Y_n) \mid \bar{Y}, S_Y^2] \\ &= E\Big[\exp\Big(\bar{Y} + \frac{Y_n - \bar{Y}}{S_Y} S_Y\Big) \mid \bar{Y}, S_Y^2\Big] \\ &= E[\exp(\bar{Y}) \mid \bar{Y}, S_Y^2] E\Big[\exp\Big(-\sqrt{\frac{n-1}{n}} S_Y U\Big) \mid \bar{Y}, S_Y^2\Big] \\ &= \exp(\bar{Y}) E\Big[\exp\Big(-\sqrt{\frac{n-1}{n}} S_Y U\Big)\Big]. \end{aligned}$$

Expanding $E[\exp(-\sqrt{\frac{n-1}{n}}S_Y U)]$ in infinite series and applying the 2nd part of Lemma,

$$E\Big[\exp\Big(-\sqrt{\frac{n-1}{n}}S_Y U\Big)\Big] = \sum_{m=0}^{\infty} \frac{(-\sqrt{\frac{n-1}{n}}S_Y)^{2m}}{(2m)!} E(U^{2m}).$$

Since U^2 has Beta distribution with parameters $\frac{1}{2}$ and $\frac{n-2}{2}$,

$$E(U^{2m}) = \frac{\Gamma(\frac{n-1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+\frac{n-1}{2})}.$$

Therefore,

$$\begin{aligned}
&E\Big[\exp(-\sqrt{\frac{n-1}{n}}S_Y U)\Big] \\
&= \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(\frac{n-1}{n}S_Y^2)^m}{(2m)!} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{n-1}{2})} \\
&= 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{n-1}{2})\Gamma(m+\frac{1}{2})(\frac{1}{2n}S_Y^2)^m 2^m (n-1)^m}{\sqrt{\pi}(2m)!\Gamma(m+\frac{n-1}{2})} \\
&= 1 + \sum_{m=1}^{\infty} (\frac{1}{2n}S_Y^2)^m \frac{1}{m!} \frac{(n-1)^m}{(n-1)(n+1)\cdots(n+2m-3)} \\
&= f(\frac{1}{2n}S_Y^2).
\end{aligned}$$

Hence $E[\exp(Y_n)\,|\,\bar{Y}, S_Y^2] = \hat{\theta}_{AM}$.

Next, we consider the UMVU estimator of η^2. We can prove easily that $E[\exp(2Y_n) - \exp(Y_n + Y_{n-1})] = \eta^2$. Therefore, by the Rao-Blackwell theorem, $E[\exp(2Y_n) - \exp(Y_n + Y_{n-1})\,|\,\bar{Y}, S_Y^2]$ is a UMVU estimator of η^2. Now we follow the same process as in finding the UMVU estimator of θ. Then we have

$$\begin{aligned}
&E\Big[\exp(2Y_n)\,|\,\bar{Y}, S_Y^2\Big] \\
&\quad = E[\exp(2\bar{Y} - 2\sqrt{\frac{n-1}{n}}S_Y U)\,|\,\bar{Y}, S_Y^2] \\
&\quad = \exp(2\bar{Y})E\Big[\exp\Big(-2\sqrt{\frac{n-1}{n}}S_Y U\Big)\Big]
\end{aligned}$$

$$= \exp(2\bar{Y})\frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{[\frac{4(n-1)}{n}S_Y^2]^m}{(2m)!}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{n-1}{2})}$$
$$= \exp(2\bar{Y})\Big\{1+\sum_{m=1}^{\infty}(\frac{2}{n}S_Y^2)^m\frac{1}{m!}\frac{(n-1)^m}{(n-1)(n+1)\cdots(n+2m-3)}\Big\}$$
$$= \exp(2\bar{Y})f(\frac{2}{n}S_Y^2).$$

On the other hand

$$\begin{aligned} Y_n + Y_{n-1} - 2\bar{Y} &= -\frac{(n-2)}{\sqrt{n(n-1)}}Z_n - \sqrt{\frac{n-2}{n-1}}Z_{n-1} \\ &= aZ_n + bZ_{n-1}, \end{aligned}$$

where $a = -\frac{(n-2)}{\sqrt{n(n-1)}}$ and $b = -\sqrt{\frac{n-2}{n-1}}$. Let $V = \frac{(Y_n+Y_{n-1}-2\bar{Y})}{S_Y} = \frac{(aZ_n+bZ_{n-1})}{S_Y}$. Then, by Basu's theorem, V is independent of $\bar{Y}$ and S_Y^2. Hence, we have

$$E[\exp(Y_n+Y_{n-1})\,|\,\bar{Y},S_Y^2] = \exp(2\bar{Y})E[\exp(S_YV)\,|\,\bar{Y},S_Y^2].$$

Now, consider the orthogonal transformation such that

$$Z_2^* = \frac{(aZ_n+bZ_{n-1})}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sum_{i=2}^{n} Z_i^2 = \sum_{i=2}^{n} Z_i^{*2}.$$

Then Z_2^*, $\cdots$, Z_n^* are i.i.d. $N(0,1)$. Also $V = \sqrt{a^2+b^2}Z_2^*/(\sum_{i=2}^n Z_i^{*2})^{\frac{1}{2}}$. Hence we have

$$\begin{aligned} &E[\exp(S_YV)\,|\,\bar{Y},S_Y^2] \\ &\quad = E[\exp(S_YV)] \\ &\quad = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{[\frac{2(n-2)}{n}S_Y^2]^m}{(2m)!}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{n-1}{2})} \\ &\quad = 1+\sum_{m=1}^{\infty}(\frac{n-2}{n(n-1)}S_Y^2)\frac{1}{m!}\frac{(n-1)^m}{(n-1)(n+1)\cdots(n+2m-3)} \\ &\quad = f(\frac{n-2}{n(n-1)}S_Y^2). \end{aligned}$$

Therefore $E[\exp(2Y_n) - \exp(Y_n+Y_{n-1})\,|\,\bar{Y},S_Y^2]$ is UMVU estimator of η^2.

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