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Tail Probability Approximations for the Ratio of two Independent Sequences of Random Variables [1]

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Abstract

In this paper, we study the saddlepoint approximations for the ratio of two independent sequences of random variables. In Section 2, we review the saddlepoint approximation to the probability density function. In section 3, we derive an saddlepoint approximation formular for the tail probability by following Daniels'(1987) method. In Section 4, we represent a numerical example which shows that the errors are small even for small sample size.

Key Words and Phrases: Saddlepoint approximation, Probability density function, Tail probability, Fourier inversion.

1. Introduction

In probability theory, one of the most important problems is to find the probability density function and the tail probability of some statistics. And it is often required to approximate the distribution of some statistics whose distribution can not be exactly obtained. When the first few moments are known, a common procedure is to fit the law of the Edgeworth type having the same moments as far as they are given (Edgeworth(1905),Wallace(1958)). This method is often satisfactory in practice, but can assume negative values in some tail regions of distribution.

Daniels(1954) introduced a new type of idea into statistics by applying saddlepoint techniques to derive a very accurate approximation to the probability density function of $\bar{X}$. He showed that the error incurred by using the saddlepoint approximation method is $O(n^{-1})$ as against the more usual $O(n^{-\frac{1}{2}})$ associated with the normal approximation. Moreover, he showed that the relative error of the approximation is uniformly $O(n^{-1})$ over the whole range of the random variable in an

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important class of cases. For reviews of saddlepoint approximations, see Reid (1988) and Field and Ronchetti (1990).

In this paper, we study the saddlepoint approximations for the ratio of independent sequences of random variables. Cho(1996) derived an saddlepoint approximation formula for the ratio $R_n = \frac{U_n}{S_n}$ where $\{U_n, n \geq 1\}$, $\{S_n > 0, n \geq 1\}$ are two independent sequences of random variables. This result can be applied to find the approximate density function of statistics such as F distribution with degree of freedom (n, n)(denote by $F(n, n)$). But this result can not be applied to find the approximate probability of ratio statistics such as $F(n_1, n_2)$, where $n_1 \neq n_2$.

In this paper we study for the ratio $R_{n1,n2} = \frac{U_{n1}}{S_{n2}}$, where $\{U_{n1}, n_1 \geq 1\}$, $\{S_{n2} > 0, n_2 \geq 1\}$ are two independent sequences of random variables using the similar method as in Daniels(1987). In section 2, we review the saddlepoint approximation to the density. And in Section 3, we derive the tailprobability approximation formular for the ratio of two independent sequences of random variables. In section 4, we represent a numerical example which shows the errors are small even for the small size. Let $\{U_{n1}, n_1 \geq 1\}, \{S_{n2} > 0, n_2 \geq 1\}$ be independent sequences of random variables with absolutely continuous distribution functions F_{n1}, F_{n2} respectively. Denote $\phi_{n1}(t) = E\{\exp(tU_{n1})\}$ and $\phi_{n2}(t) = E\{\exp(tS_{n2})\}$ be the moment generating functions of U_{n1} and S_{n2}, respectively. And let $\psi_{n1}(t) = (1/n_1)log\phi_{n2}(t)$, and $\psi_{n2}(t) = (1/n_2)log\phi_{n2}(t)$ be their cumulant generating function. Assume that $\phi_{n1}(t)$ and $\psi_{n1}(t)$ exist for real t in some interval (t_1, t_2) containing 0 and that $\phi_{n2}(t)$ and $\psi_{n2}(t)$ exist for real t in some interval (t_3, t_4) containing 0.

2. Saddlepoint Approximation

The fact that the integrand in Fourier inversion the density of $R_{n1,n2}$ is of the form $\exp[n_2\{\Psi_{n1,n2}(z)\}]$ is the starting point to derive the saddlepoint approximations for $R_{n1,n2}$.

Let $H_{n1,n2}$ be the distribution function of $R_{n1,n2}$. Then

$$H_{n1,n2}(r) = P_r(R_{n1,n2} \leq r) = \int_0^{+\infty} F_{n1}(ry)\, dF_{n2}(y). \tag{1}$$

And the p.d.f. $h_{n1,n2}$ of $R_{n1,n2}$ is given by

$$h_{n1,n2}(r) = \int_0^{+\infty} y f_{n1}(ry)\, dF_{n2}(y), \tag{2}$$

where f_{n1} is the p.d.f. of U_{n1}.

The characteristic function of $R_{n1,n2}$ is given by

$$\begin{aligned} \hat{h}_{n1,n2}(t) &= \int_{-\infty}^{+\infty} e^{itr} \int_0^{+\infty} y f_{n1}(ry)\, dF_{n2}(y)\, dr \\ &= \int_{-\infty}^{+\infty} \phi_{n1}(\frac{it}{y})\, dF_{n2}(y). \end{aligned} \tag{3}$$

Using the Fourier inversion formula, the p.d.f. $h_{n1,n2}$ is given by

$$\begin{aligned} h_{n1,n2}(r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itr}\hat{h}_{n1,n2}(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\{\int_{0}^{+\infty}\phi_{n1}(\frac{it}{y})dF_{n2}(y)\}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi_{n1}(is)\{\int_{0}^{+\infty}e^{-isyr}\cdot ydF_{n2}(y)\}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi_{n1}(it)\phi'_{n2}(-irt)dt \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi_{n1}(z)\phi'_{n2}(-rz)dz \\ &= \frac{n_2}{2\pi i}\int_{c-i\infty}^{c+i\infty} exp[n_2\{\frac{n_1}{n_2}\psi_{n1}(z)+\psi_{n2}(-rz)\}]\times\psi'_{n2}(-rz)dz \end{aligned} \tag{4}$$

where c is any real number in the interval where the moment generating function exists.

When n_1, n_2 is large, an approximation is found by passing the path of integration through a saddlepoint τ of the exponential part of integrand given by $n_1\psi'_{n1}(\tau) - rn_2\ \psi'_{2n}(-r\tau) = 0$. We choose c to be τ.

On the contour near τ, we have

$$\begin{aligned} n_2\{\frac{n_1}{n_2}\psi_{n1}(z)+\psi_{n2}(-rz)\} &= n_2\ \Psi_{n1,n2}(z)(say) \\ &= n_2\{\Psi_{n1,n2}(\tau)+\frac{\Psi''(\tau)}{2}(z-\tau)^2+\frac{\Psi^{(3)}(\tau)}{6}(z-\tau)^3+\cdots\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \psi'_{n2}(-rz) &= \psi'_{n2}(-r\tau) - r\psi^{(2)}_{n2}(-r\tau)(z-\tau) \\ &+\frac{r^2}{2}\psi^{(3)}_{n2}(-r\tau)(z-\tau)^2 - \frac{r^3}{6}\psi^{(4)}_{n2}(-r\tau)(z-\tau)^3+\cdots. \end{aligned} \tag{6}$$

Let $\sqrt{n_2\Psi''_{n1,n2}(\tau)}(z-\tau) = iy$ and expanding the integrand in (4) near τ, we have

$$\begin{aligned} h_{n1,n2}(r) &= \left(\frac{n_2}{\hat{\Psi}''_{n1,n2}}\right)^{\frac{1}{2}}\frac{\exp[n_1\{\psi_{n1}(\tau)+n_2\psi_{n2}(-r\tau)\}]}{2\pi}\psi'_{n2}(-r\tau) \\ &\times\int_{-\infty}^{+\infty}\exp\left(\frac{y^2}{2}\right)(1-\frac{\lambda_3}{6\sqrt{n_2}}iy^3+\frac{\lambda_4}{24n_2}y^4-\frac{\lambda_3^2}{72n_2}y^6+\cdots) \end{aligned}$$

$$\times\{1-\frac{r\psi_{n2}^{(2)}(-r\tau)}{\psi'_{n2}(-r\tau)}\frac{iy}{\sqrt{n}_2(\hat{\Psi}''_{n1,n2})^{\frac{1}{2}}}-\frac{r^2\psi_{n2}^{(3)}(-r\tau)}{2n_2\psi'_{n2}(-r\tau)\hat{\Psi}''_{n1,n2}}y^2$$

$$+\frac{r^3\psi_{n2}^{(4)}(-r\tau)}{6n_2\sqrt{n}_2\psi'_{n2}(-r\tau)(\hat{\Psi}''_{n1,n2})^{\frac{3}{2}}}y^3+\cdots)dy\}$$

$$=\tilde{h}_{n1,n2}(r)[1+\frac{3}{n_2}\left\{\frac{\lambda_4}{24}-\frac{r\psi_{n2}^{(2)}(-r\tau)\lambda_3}{6\psi'_{n2}(-r\tau)(\hat{\Psi}''_{n1,n2})^{\frac{1}{2}}}\right\}-\frac{15\lambda_3^2}{72n_2}$$

$$-\frac{r^2\psi_{n2}^{(3)}(-r\tau)}{2n_2\psi'_{n2}(-r\tau)\hat{\Psi}''_{n1,n2}}+\cdots], \tag{7}$$

where

$$\tilde{h}_{n1,n2}(r)=\frac{\sqrt{n_2}\psi'_{n2}(-r\tau)\exp[n_1\psi_{n1}(\tau)+n_2\psi_{n2}(-r\tau)]}{\sqrt{2\pi\{\frac{n_1}{n_2}\psi''_{n1}(\tau)+r^2\psi''_{n2}(-r\tau)\}}} \tag{8}$$

and

$$\hat{\Psi}^{(j)}_{n1,n2}=\Psi^{(j)}_{n1,n2}(\tau),\ \lambda_j=\frac{\hat{\Psi}^{(j)}_{n1,n2}}{(\hat{\Psi}''_{n1,n2})^{j/2}}.$$

We call $\tilde{h}_{n1,n2}(r)$ the saddlepoint approximation to $h_{n1,n2}(r)$.

Remark. In the above if $n_1=n_2=n$,then (8) becomes as follows.

$$\tilde{h}_{n,n}(r)=\frac{\sqrt{n}\psi'_{n2}(-r\tau)\exp[n(\psi_{n1}(\tau)+\psi_{n2}(-r\tau))]}{\sqrt{2\pi\{\psi''_{n1}(\tau)+r^2\psi''_{n2}(-r\tau)\}}}. \tag{9}$$

3. Tail Probability Approximations

In this section, We are interested in the tail probability of $R_{n1,n2}$, i.e., $P_r(R_{n1,n2}\geq r)=\bar{H}_{n1,n2}(r)$ and derive an approximation formular for it. The tail probability can be approximated by integrating the saddlepoint approximation $\tilde{h}_{n1,n2}(r)$ numerically, i.e.,

$$\tilde{\bar{H}}_{n1,n2}(r)=\int_r^\infty \tilde{h}_{n1,n2}(y)dy. \tag{10}$$

To obtain the explicit approximate formula for the tail probability, we must consider the next inversion formula.

$$\bar{H}_{n1,n2}(r)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[n_1\psi_{n1}(z)+n_2\psi_{n2}(-rz)\}]\frac{dz}{z}\quad(c>0). \tag{11}$$

The above relation (11) for the tail probability of $R_{n1,n2}$ is obtained as follows. Since the p.d.f of $R_{n1,n2}$ is

$$h_{n1,n2}(r) = \frac{n_2}{2\pi}\int_{-\infty}^{+\infty} \exp[n_1\psi_{1n}(c+it)+n_2\psi_{n2}(-r(c+it))\}]\times\psi'_{n2}\{-r(c+it)\}dt \quad (12)$$

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= \frac{n_2}{2\pi}\int_r^{+\infty}\int_{-\infty}^{+\infty} \exp[n_2\{\frac{n_1}{n_2}\psi_{n1}(c+it)+\psi_{n2}(-r(c+it))\}] \\ &\quad \times\psi'_{n2}\{-r(c+it)\}dtdr \qquad (13)\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{n_2\{\frac{n_1}{n_2}\psi_{n1}(c+it)\}[-\frac{1}{c+it} \\ &\quad \exp\{n_2\{\frac{n_1}{n_2}\psi_{n2}(-r(c+it))\}|_r^{\infty}]dt\end{aligned}$$

Since $\exp\{n_2\{\psi_{n2}(-r(c+it))\} = \phi_{n2}\{-r(c+it)\} = \int_0^\infty \exp\{-r(c+it)x\}\times dF_{n2}(x)$ converges to zero as $r\to\infty$ and $c>0$, the relation (13) is obtained. As in Daniels (1987), expanding only $\Psi_{n1,n2}(z) = \frac{n_1}{n_2}\psi_{n1}(z)+\psi_{n2}(-rz)$ and integrating, we have

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= \exp(n_2\hat{\Psi}_{n1,n2})\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\{\frac{1}{2}\hat{\Psi}''_{n1,n2}(z-\tau)^2\} \\ &\quad\times\{1+\frac{1}{6}\hat{\Psi}^{(3)}_{n1,n2}(z-\tau)^3+\frac{1}{24}n\hat{\Psi}^{(4)}_{n1,n2}(z-\tau)^4 \\ &\quad+\frac{1}{72}n_2^2(\hat{\Psi}^{(3)}_{n1,n2})^2(z-\tau)^6+\cdots\}\frac{dz}{z} \\ &= \exp(n_2\hat{\Psi}_{n1,n2}+\frac{1}{2}\hat{u}^2)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\frac{1}{2}u^2-\hat{u}u) \qquad (14)\\ &\quad\times[1+\frac{\lambda_3}{6\sqrt{n}}(u-\hat{u})^3+\frac{1}{n_2}\{\frac{\lambda_4}{24}(u-\hat{u})^4+\frac{\lambda_3^2}{72}(u-\hat{u})^6\}+\cdots]\frac{du}{u},\end{aligned}$$

where $u = z(n_2\hat{\Psi}''_{n1,n2})^{\frac{1}{2}}$ and $\lambda_j = \hat{\Psi}^{(j)}_{n1,n2}/(\hat{\Psi}''_{n1,n2})^{j/2}$, $j\ge 3$. The above tail probability $\bar{H}_{n1,n2}(r)$ can be found from the fact that $I_r = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\frac{1}{2}u^2-\hat{u}u)(u-\hat{u})^r\, du/u$ satisfies the recurrence relations :

$$\begin{aligned}I_{2m} &= (-\hat{u})I_{2m-1},\ I_{2m+1} = (-1)^m a_m\phi(\hat{u})-\hat{u}I_{2m} \\ \text{with } I_0 &= 1-\Phi(\hat{u}),\ a_0 = 1 \text{ and } a_m = 1\times\cdots\times(2m-1) \qquad (15)\end{aligned}$$

Repeated applications of (15) lead to the explicit formula

$$I_r = (-\hat{u})^r\{1-\Phi(\hat{u})\}+(-1)^{r-1}\,\phi(\hat{u})\sum_{m=0}^{[\frac{1}{2}(r-1)]}(-1)^m\,a_m\,\hat{u}^{r-2m-1}. \qquad (16)$$

In (14), $I_3 = -\hat{u}^3\{1-\Phi(\hat{u})\} + (\hat{u}^2-1)\phi(\hat{u})$ gives the following formula.

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= \exp(n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2) \times [\{1-\Phi(\hat{u})\}(1-\frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}}) \\ &+\phi(\hat{u})\frac{\lambda_3}{6\sqrt{n_2}} \times (\hat{u}^2-1)] \times \{1+O(n_2^{-1})\} \\ &= \tilde{\bar{H}}^1_{n1,n2}(r)\{1+O(n_2^{-1})\} \ \text{(say)}. \end{aligned} \tag{17}$$

And substitution of I_4 and I_6 in (14) gives the following formula;

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= \exp(n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2)[\{1-\Phi(\hat{u})\}\{1-\frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}} + \frac{1}{n_2}\left(\frac{\lambda_4\hat{u}^4}{24} + \frac{\lambda_3^2\hat{u}^6}{72}\right)\} \\ &+\phi(\hat{u})\{\frac{\lambda_3}{6\sqrt{n_2}}(\hat{u}^2-1) - \frac{1}{n_2}(\frac{\lambda_4}{24}(\hat{u}^3-\hat{u}) + \frac{\lambda_3^2}{72}(\hat{u}^5-\hat{u}^3+3\hat{u}))\}] \\ &\times\{1+O(n_2^{-\frac{3}{2}})\} \\ &= \tilde{\bar{H}}^2_{n1,n2}(r)\{1+O(n_2^{-\frac{3}{2}})\} \ \text{(say)}. \end{aligned} \tag{18}$$

When $\tau \le 0$, we can also use the following inversion formula

$$\bar{H}_{n1,n2}(r) = h(-\tau) + \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp\{n_2\ \Psi_{n1,n2}(z)\}\frac{dz}{z}, \tag{19}$$

where $h(x) = 0, \frac{1}{2}, 1$, when $x < 0, = 0, > 0$, respectively. See Daniels(1987) p 43. So, (17) can be replced by

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= h(-\hat{u}) + \exp(n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2) \times [\{h(\hat{u})-\Phi(\hat{u})\}\{1-\frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}}\} \\ &+\phi(\hat{u})\frac{\lambda_3}{6\sqrt{n_2}} \times (\hat{u}^2-1)] \times \{1+O(n_2^{-1})\} \\ &= \tilde{\bar{H}}^1_{n1,n2}(r) \times \{1+O(n_2^{-1})\} \ \text{(say)}. \end{aligned} \tag{20}$$

And (18) can be replaced by

$$\begin{aligned}\bar{H}_{n1,n2}(r) &= h(-\hat{u}) + \exp\{n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2\}[\{h(\hat{u})-\Phi(\hat{u})\}\{1-\frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}} \\ &+\frac{1}{n_2}\left(\frac{\lambda_4\hat{u}^4}{24} + \frac{\lambda_3^2\hat{u}^6}{72}\right)\} + \phi(\hat{u})[\frac{\lambda_3}{6\sqrt{n_2}}(\hat{u}^2-1) - \frac{1}{n_2}\{\frac{\lambda_4}{24}(\hat{u}^3-\hat{u}) \\ &+\frac{\lambda_3^2}{72}(\hat{u}^5-\hat{u}^3+3\hat{u})\}]] \times \{1+O(n_2^{-\frac{3}{2}})\} \\ &= \tilde{\bar{H}}^2_{n1,n2}(r)\{1+O(n_2^{-\frac{3}{2}})\} \ \text{(say)}. \end{aligned} \tag{21}$$

4. Numerical Example

In this section, we present an example to show that the errors of our saddlepoint approximation formular are small.

Example

Assume that $\{U_{n1}\}$ and $\{S_{n2}\}$ have $\chi^2(n_1)$, $\chi^2(n_2)$distribution respectively and are independent. Then the exact p.d.f. of $R_{n_1,n_2} = U_{n1}/S_{n2}$ can be obtained from the fact that $\frac{n_2}{n_1}R_{n_1,n_2}$ follows F-distribution with (n_1, n_2) degrees of freedom(Denote,$F(n_1, n_2)$). So the exact probability density function of $R_{n1,n2}$ is given by

$$h_{n1,n2}(r) = \frac{\Gamma(\frac{n_1+n_2}{2})r^{\frac{n_1}{2}-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})(1+r)^{\frac{n_1+n_2}{2}}}. \qquad (22)$$

By definition, we have

$$\begin{aligned} \phi_{ni}(z) &= (1-2z)^{-\frac{n_i}{2}}, \psi_{ni}(z) = -\frac{1}{2}\, log(1-2z), \\ \psi'_{ni}(z) &= (1-2z)^{-1} \text{ and } \psi_{ni}^{(2)} = 2(1-2z)^{-2} \ (i=1,2). \end{aligned}$$

So the saddlepoint equation becomes $n_1\psi'_{n1}(\tau) - n_2 r\psi'_{n2}(-r\tau) = 0$ and the saddlepoint $\tau = \frac{n_2 r - n_1}{2r(n_1+n_2)}$. Therefore the saddlepoint approximation to the p.d.f. of R_{n_1,n_2} is given by

$$\begin{aligned} \tilde{h}_{n1,n2}(r) &= \frac{\sqrt{n}_2\psi'_{n2}(-r\tau)\exp[n_1\psi_{n1}(\tau) + n_2\psi_{n2}(-r\tau)]}{\sqrt{2\pi\{\frac{n_1}{n_2}\psi''_{n1}(\tau) + r^2\psi''_{n2}(-r\tau)\}}} \\ &= \frac{(\frac{n_1+n_2}{n_1})^{\frac{n_1-1}{2}}(\frac{n_1+n_2}{n_2})^{\frac{n_2}{2}}\ r^{\frac{n_1}{2}-1}}{2\sqrt{\pi}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}(1+r)^{\frac{n_1+n_2}{2}}}. \qquad (23) \end{aligned}$$

Note that the ratio $B_{n1,n2}(r) = \frac{\tilde{h}_{n1,n2}(r)}{h_{n1,n2}(r)} = \frac{\Gamma(\frac{n_1-n_2}{2})(\frac{n_1-n_2}{n_1})^{\frac{n_1}{2}}(\frac{n_1-n_2}{n_2})^{\frac{n_2}{2}}}{2\sqrt{\pi}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ does not depend on r.

We can obtain the exact tail probability by integrating $h_{n1,n2}(r)$ from r to ∞, i.e.,

$$\bar{H}_{n1,n2}(r) = \int_r^\infty \frac{\Gamma(\frac{n_1+n_2}{2})r^{\frac{n_1}{2}-1}}{\Gamma^2(n_1/2)\Gamma^2(n_2/2)\ (1+r)^{n_1+n_2}}\, dx. \qquad (24)$$

From (20) and (21) we obtain two approximation formulae for the tail probability of $R_{n1,n2}$, i.e.,

$$\tilde{\tilde{H}}^1_{n1,n2}(r) = h(-\hat{u}) + \exp(n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2) \times [\{h(\hat{u}) - \Phi(\hat{u})\}\{1 - \frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}}\} + \phi(\hat{u})\frac{\lambda_3}{6\sqrt{n_2}} \times (\hat{u}^2 - 1)] \quad (25)$$

and

$$\tilde{\tilde{H}}^2_{n1,n2}(r) = h(-\hat{u}) + \exp\{n_2\hat{\Psi}_{n1,n2} + \frac{1}{2}\hat{u}^2\}[\{h(\hat{u}) - \Phi(\hat{u})\}\{1 - \frac{\lambda_3\hat{u}^3}{6\sqrt{n_2}} + \frac{1}{n_2}\left(\frac{\lambda_4\hat{u}^4}{24} + \frac{\lambda_3^2\hat{u}^6}{72}\right)\} + \phi(\hat{u})[\frac{\lambda_3}{6\sqrt{n_2}}(\hat{u}^2 - 1) - \frac{1}{n_2}\{\frac{\lambda_4}{24}(\hat{u}^3 - \hat{u}) + \frac{\lambda_3^2}{72}(\hat{u}^5 - \hat{u}^3 + 3\hat{u})\}]] \quad (26)$$

where,

$$\hat{\Psi}''_{n1,n2} = \frac{2r^2(n_1+n_2)^2}{(r+1)^2}(\frac{1}{n_1n_2} + \frac{1}{n_2^2}), \hat{\Psi}^{(3)}_{n1,n2} = \frac{8r^3(n_1+n_2)^3}{(1+r)^3}(\frac{n_2^2 - n_1^2}{n_1^2n_2^3}),$$

$$\hat{\Psi}^{(4)}_{n1,n2} = \frac{48r^4(n_1+n_2)^4}{(1+r)^4}(\frac{1}{n_1^3n_2} + \frac{1}{n_2^4}), \lambda_3 = \frac{\hat{\Psi}^{(3)}}{(\hat{\Psi}^{(2)})^{\frac{3}{2}}}, \lambda_4 = \frac{\hat{\Psi}^{(4)}}{(\hat{\Psi}^{(2)})^2}$$

$$\hat{u} = \hat{\tau}(n_2\hat{\Psi}^{(2)})^{\frac{1}{2}}.$$

Table 4.3 and Table 4.4 represent the results of (24), (25), (26) when $(n_1, n_2) = (3, 4), (n_1, n_2) = (6, 9), (n_1, n_2) = (8, 16), (n_1, n_2) = (15, 16)$ with increasing r by 0.2. The exact values of tail probability of $F-$ distributions are computed from IMSL. Using the following relation

$$P(\frac{\chi^2(n_1)}{\chi^2(n_2)} \geq r) = P(F(n_1, n_2) \geq r \cdot \frac{n_2}{n_1}) \quad (27)$$

we can obtain the exact values of (24). From those tables we can see that the errors of saddlepoint approximation formular are small in this Example.

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Table 4.1. *Exact probability from (24) and approximations* $\tilde{\tilde{H}}^{1}_{n_1,n_2}$ *from (25),* $\tilde{\tilde{H}}^{2}_{n_1,n_2}$ *from (26) for* $P_r\{\chi(3)/\chi(4) \geq r\}$ *with increasing r by 0.2*

r	Exact	$\tilde{\tilde{H}}^{1}_{n_1,n_2}$	$\tilde{\tilde{H}}^{2}_{n_1,n_2}$
.1	0.935212	0.9436568	0.9385280
.3	0.761229	0.7847526	0.7729792
.5	0.615100	0.6419749	0.6330253
.7	0.502636	0.5253373	0.5234997
.9	0.416610	0.4323735	0.4371483
1.0	0.381282	0.3938130	0.4009916
1.1	0.350105	0.3598317	0.3688263
1.3	0.297932	0.3033216	0.3145565
1.5	0.256387	0.2588199	0.2710241
1.7	0.222835	0.2232898	0.2357085
1.9	0.195385	0.1945303	0.2067342
2.1	0.172663	0.1709510	0.1827072
2.3	0.153654	0.1513915	0.1625837
2.5	0.137598	0.1349941	0.1455751
2.7	0.123918	0.1211159	0.1310781
2.9	0.112171	0.1092682	0.1186268
3.1	0.102011	0.0990746	0.1078571
3.3	0.093166	0.0902416	0.09848151
3.5	0.085419	0.0825381	0.09027103
3.7	0.078597	0.0757795	0.08304149
3.9	0.072558	0.0698177	0.07664327
4.1	0.067188	0.0645322	0.07095426
4.3	0.062391	0.0598247	0.06587392
4.5	0.058089	0.0556138	0.06131862
4.7	0.054216	0.0518322	0.05721876
4.9	0.050717	0.0484234	0.05351567
5.1	0.047545	0.0453401	0.05015992
5.3	0.044662	0.0425421	0.04710957
5.5	0.042033	0.0399952	0.04432869
5.7	0.039629	0.0376704	0.04178647
5.9	0.037425	0.0355424	0.03945648

Table 4.2. *Exact probability from (24) and approximations* $\tilde{\tilde{H}}^1_{n_1,n_2}$ *from (25),* $\tilde{\tilde{H}}^2_{n_1,n_2}$ *from (26) for* $P_r\{\chi(6)/\chi(9) \geq r\}$ *with increasing r by 0.2*

r	Exact	$\tilde{\tilde{H}}^1_{n_1,n_2}$	$\tilde{\tilde{H}}^2_{n_1,n_2}$
.1	0.984242	0.9857668	0.9849284
.3	0.828350	0.8418876	0.8363347
.5	0.624973	0.6465091	0.6418479
.7	0.454657	0.4737881	0.4747539
.9	0.328902	0.3410333	0.3460508
1.1	0.239597	0.2465188	0.2528427
1.3	0.176648	0.1804239	0.1866146
1.5	0.132036	0.1340091	0.1395162
1.7	0.100077	0.1010322	0.1057262
1.9	0.076883	0.0772696	0.0811923
2.1	0.059822	0.0598949	0.0631459
2.3	0.047104	0.0470086	0.0496968
2.5	0.037501	0.0373210	0.0395462
2.7	0.030162	0.0299450	0.0317924
2.9	0.024490	0.0242621	0.0258020
3.1	0.020059	0.0198352	0.0211247
3.3	0.016563	0.0163512	0.0174362
3.5	0.013779	0.0135831	0.0145005
3.7	0.011544	0.0113642	0.0121437
3.9	0.009734	0.0095710	0.0102365
4.1	0.008257	0.0081105	0.0086813
4.3	0.007045	0.0069125	0.0074043
4.5	0.006042	0.0059233	0.0063487
4.7	0.005208	0.0051012	0.0054708
4.9	0.004510	0.0044140	0.0047364
5.1	0.003922	0.0038364	0.0041186
5.3	0.003425	0.0033483	0.0035962
5.5	0.003003	0.0029339	0.0031524
5.7	0.002643	0.0025803	0.0027736
5.9	0.002334	0.0022774	0.0024488

Table 4.3 *Exact probability from (24) and approximations* $\tilde{\bar{H}}^{1}_{n_1,n_2}$ *from (25),* $\tilde{\bar{H}}^{2}_{n_1,n_2}$ *from (26) for* $P_r\{\chi(2)/\chi(16) \geq r\}$ *with increasing r by 0.2*

r	Exact	$\tilde{\bar{H}}^{1}_{n_1,n_2}$	$\tilde{\bar{H}}^{2}_{n_1,n_2}$
.1	0.466507	0.482130	0.475097
.3	0.122589	0.119546	0.134548
.5	0.039018	0.037884	0.439543
.7	0.014335	0.013958	0.161470
.9	0.005888	0.574699	0.006677
1.1	0.002644	0.002584	0.002997
1.3	0.001277	0.001249	0.001436
1.5	0.000655	0.000641	0.000741
1.7	0.000354	0.000346	0.000400
1.9	0.000200	0.000195	0.000225
2.1	0.000117	0.000114	0.000132
2.3	0.000071	0.000069	0.000079
2.5	0.000044	0.000043	0.000050
2.7	0.000028	0.000027	0.000031
2.9	0.000019	0.000018	0.000021
3.1	0.000013	0.000012	0.000014
3.3	0.000009	0.000008	0.000009
3.5	0.000006	0.000005	0.000006
3.7	0.000004	0.000004	0.000004
3.9	0.000003	0.000002	0.000003
4.1	0.000002	0.000002	0.000002
4.3	0.000002	0.000001	0.000001
4.5	0.000001	0.000001	0.000001
4.7	0.000001	0.000008	0.000000
4.9	0.000001	0.000000	0.000000
5.1	0.000001	0.000000	0.000000
5.3	0.000000	0.000000	0.000000
5.5	0.000000	0.000000	0.000000
5.7	0.000000	0.000000	0.000000
5.9	0.000000	0.000000	0.000000

Table 4.4. *Exact probability from (24) and approximations* $\tilde{\tilde{H}}^{1}_{n_1,n_2}$ *from (25),* $\tilde{\tilde{H}}^{2}_{n_1,n_2}$ *from (26) for* $P_r\{\chi(8)/\chi(16) \geq r\}$ *with increasing r by 0.2*

r	Exact	$\tilde{\tilde{H}}^{1}_{n_1,n_2}$	$\tilde{\tilde{H}}^{2}_{n_1,n_2}$
.1	0.986640	0.987441	0.986946
.3	0.764719	0.774733	0.770107
.5	0.472557	0.483377	0.483377
.7	0.269156	0.273260	0.277459
.9	0.150858	0.151768	0.155930
1.1	0.085437	0.085347	0.088388
1.3	0.049407	0.049096	0.051122
1.5	0.029281	0.028981	0.030294
1.7	0.017796	0.017559	0.018406
1.9	0.011083	0.010908	0.011517
2.1	0.007063	0.006938	0.007339
2.3	0.004600	0.004510	0.004753
2.5	0.003056	0.002992	0.003173
2.7	0.002068	0.002022	0.002135
2.9	0.001424	0.001391	0.001478
3.1	0.000996	0.000972	0.001027
3.3	0.000707	0.000689	0.000729
3.5	0.000508	0.000495	0.000524
3.7	0.000370	0.000360	0.000384
3.9	0.000273	0.000265	0.000281
4.1	0.000203	0.000197	0.000209
4.3	0.000153	0.000148	0.000157
4.5	0.000116	0.000113	0.000119
4.7	0.000089	0.000086	0.000092
4.9	0.000069	0.000067	0.000071
5.1	0.000054	0.000052	0.000055
5.3	0.000042	0.000040	0.000043
5.5	0.000033	0.000032	0.000034
5.7	0.000027	0.000025	0.000027
5.9	0.000021	0.000020	0.000022

Table 4.5. *Exact probability from (24) and approximations* $\tilde{\bar{H}}^{1}_{n_1,n_2}$ *from (25),* $\tilde{\bar{H}}^{2}_{n_1,n_2}$ *from (26) for* $P_r\{\chi(15)/\chi(16) \geq r\}$ *with increasing r by 0.2*

r	Exact	$\tilde{\bar{H}}^{1}_{n_1,n_2}$	$\tilde{\bar{H}}^{2}_{n_1,n_2}$
.1	0.999959	0.999960	0.999959
.3	0.983563	0.984103	0.983667
.5	0.884522	0.887291	0.885230
.7	0.711648	0.715782	0.713284
.9	0.529295	0.531983	0.531487
1.1	0.376349	0.376522	0.378261
1.3	0.261698	0.260470	0.263071
1.5	0.180456	0.178817	0.181382
1.7	0.124426	0.122855	0.125037
1.9	0.086207	0.084873	0.086609
2.1	0.060185	0.059115	0.060451
2.3	0.042405	0.041571	0.042582
2.5	0.030177	0.029536	0.030296
2.7	0.021695	0.021206	0.021777
2.9	0.015758	0.015385	0.015814
3.1	0.011561	0.011276	0.011600
3.3	0.008564	0.008346	0.008592
3.5	0.006404	0.006236	0.006424
3.7	0.004832	0.004702	0.004847
3.9	0.003678	0.003576	0.003688
4.1	0.002822	0.002743	0.002829
4.3	0.002182	0.002120	0.002188
4.5	0.001700	0.001651	0.001704
4.7	0.001334	0.001294	0.001337
4.9	0.001053	0.001022	0.001056
5.1	0.000837	0.000812	0.000839
5.3	0.000670	0.000649	0.000671
5.5	0.000539	0.000522	0.000539
5.7	0.000436	0.000422	0.000436
5.9	0.000354	0.000343	0.000355