Journal of the Korean
Data & Information Science Society
1999, Vol. 10, No. 2, pp. 341 ~ 357

Better Bootstrap Confidence Intervals for Process Incapability Index C_{pp}

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Abstract

Greenwich and Jahr-Schaffrath(1995) considered a new process incapability index(PII) C_{pp}, which modified the useful index C_{pm}^* for detecting assignable causes. The new index C_{pp} provides an uncontaminated separation between information concerning the process accuracy and precision while this kind of information separation is not available with the C_{pm}^* index.

In this paper, we will study about the index C_{pp} based on the bootstrap. First, we will prove the consistency of bootstrap deriving the bootstrap asymptotic distribution for our index C_{pp}. Moreover, with the consistency of bootstrap, we will construct six bootstrap confidence intervals and compare their performances. Some simulation results, comparison and analysis are provided. In particular, two STUD and ABC bootstrap methods perform significantly better.

Key Words and Phrases: Bootstrap, Process Incapability Index, Asymptotic Normality, Inaccuracy Index, Imprecision Index.

1. Introduction

A process capability index(PCI) is a unitless measure showing how capable a manufacturing process is of operating within its engineering specifications. If it is greater than or equal to the value, the process is considered to be capable.

On the other hand, process capability can be expressed with a process index which indicates the incapability of a process to meet its specifications. This index

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is regarded as a PCI or more precisely as a PII which is obtained from a simple *transformation* of a PCI. The PII contains the same information as the PCI and provides an uncontaminated separation between information concerning the process accuracy and precision. This kind of information separation is not available with PCI. If it is less than or equal to the value, the process is considered capable.

Even when the underlying process is assumed to be normal, the computation of PII results in the estimator being random variable possessing very complicated distribution. Thus, there exist some based on large sample properties for PCIs such as Chan et al.*(1990)* and so on. *But there exist only a few studies on the asymptotic* distributions for PIIs, notably Greenwich and Jahr-Schaffrath(1995).

However, generally the central limit theorem is not of use since it demands at least 100 observations. Moreover, since many processes can frequently be skewed or heavy-tailed in practice, an interval estimation technique that is free from the assumption of distribution is desirable.

Bootstrap sampling techniques can be successfully used to construct reasonably accurate estimates for the PCIs or PIIs when process information is limited. A bootstrap method is precisely such a technique applied for hundreds of papers for that reason. Thus, since Kim and Cho(1995) there have been many studies on *the asymptotic distributions of the estimators* for PCIs and bootstrap asymptotic distributions of those.

Only recently have lower confidence limits been introduced when the process is assumed to be normally distributed. In particular, Franklin and Wasserman(1991, 1992), Kim and Cho(1995), and Cho et al.(1999) have studied the confidence limit for the correct interpretation of PCIs. However, there are no studies yet on bootstrap asymptotic distributions as well as confidence limits for the PIIs.

In this paper, we will derive the bootstrap asymptotic distributions for our index C_{pp}, and construct six bootstrap confidence intervals for incapability index . Also, we will present the simulated results to compare their performances C_{pp} based on the consistency of the bootstrap.

2. Process Incapability Index

2.1 Process Incapability Index

Process Capability can be expressed with a process index which indicates the incapability of a process to meet its specifications. This index is regarded as a PCI or more precisely as a process incapability index(PII). It is obtained from a simple transformation of a PCI. The PII contains the same information as the PCI. Moreover, the PII provides an uncontaminated separation between information concerning the process accuracy and precision.

Assume that the characteristic variable X is normally distributed with mean, μ, and standard deviation, σ, where they are not known in practice. An allowable process spread, called process specification, generally consists of lower and upper specification limits (LSL, USL). A target value T lies somewhere between these limits, and $d = (USL - LSL)/2$(that is, half the length of the specification interval on the characteristic X of each).

Now we consider the definitions of two PIIs, C_{pg} and C_{pp}. Marcucci and Beazley(1988) suggested a PII, C_{pg}, which can be by a transformation of the index, C_{pm}.

$$\begin{aligned} C_{pg} &= C_{pm}^{-2} \\ &= \left(\frac{3\tau}{d}\right)^2 \\ &= \left\{\frac{3(\mu - T)}{d}\right\}^2 + \left(\frac{3\sigma}{d}\right)^2 \\ &= C_{ia}^{'} + C_{ip}^{'} , \end{aligned} \tag{1}$$

where $C_{ia}^{'}=\{3(\mu - T)\}^2$, $C_{ip}^{'}=\{3\sigma/d\}^2$.

They summarize some properties of C_p-type indices and their estimators. We say that the process is in control when $C_{pp} = 0$ and it provides more information than the PCI C_{pm}. The transformation allows one to separate information of the process variation from that of the departure from the process target. The subindices $C_{ia}^{'}$ and $C_{ip}^{'}$ are indices of inaccuracy and imprecision, respectively. The C_{pg} partition is additive, while the C_{pk} partition is multiplicative. If other components of the variation are known, the subindex $C_{ip}^{'}$ can be further partitioned.

Greenwich and Jahr-Schaffrath(1995) applied the transformation to C_{pm}^{*} for $T \neq M$.

$$\begin{aligned} C_{pp} &= {C_{pm}^{*}}^{-2} \\ &= \left\{\frac{3\tau}{min(USL - T, T - LSL)}\right\}^2 \\ &= \left(\frac{\mu - T}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 \\ &= C_{ia} + C_{ip} , \end{aligned} \tag{2}$$

where $D = min(\frac{USL-T}{3}, \frac{T-LSL}{3})$, $C_{ia} = \left(\frac{\mu-T}{D}\right)^2$, $C_{ip} = (\frac{\sigma}{D})^2$.

Since the denominators of subindices are identical, they provide the relative magnitudes of the contributions to the process in capability indicated by C_{pp}. So we note that $100C_{ia}/C_{pp}$ and $100C_{ip}/C_{pp}$ provide the proportions of the process incapability.

2.2 Asymptotic Distributions of PIIs

Suppose that a set of the independent random variables $X_1, X_2, ..., X_n$ has a common distribution $F(\cdot)$ with process mean μ and standard deviation σ. Only a few studies on PIIs exist as mentioned in the previous section. For asymptotic properties consider the natural plug-in estimator of C_{pp} from the equation (2) in the previous section,

$$\hat{C}_{pp} = \left\{ \frac{3\sqrt{S^2 + (\bar{X} - T)^2}}{min(USL - T,\ T - LSL)} \right\}^2 \tag{3}$$

The asymptotic distributions of the estimators for PIIs have not appeared in many studies. Now we consider the lemma which is needed to derive the asymptotic distributions. We say that X_n converges in distribution(or in law) to X if for each continuity point T of F. This is written $X_n \xrightarrow{d} X$.

Lemma If μ_4 exists and $g(u,v)$is a real valued function which is differential,

$$(a)\quad \sqrt{n}(\bar{X} - \mu, S^2 - \sigma^2) \xrightarrow{d} BN(\mathbf{0}, \mathbf{\Sigma}).$$

$$(b)\quad \sqrt{n}\left(g(\bar{X}, S^2) - g(\mu, \sigma^2)\right) \xrightarrow{d} N(0, \mathbf{d}^t\mathbf{\Sigma}\mathbf{d}),$$

$$\text{where } \mathbf{d}^t = \left(\left.\frac{\partial g(u,v)}{\partial u}\right|_{\mu,\sigma^2}, \left.\frac{\partial g(u,v)}{\partial v}\right|_{\mu,\sigma^2} \right) \neq (0,0),$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_3 - \sigma^4 \end{pmatrix}.$$

Proof See pp.72 and pp.122 of Serfling(1980).

Asymptotic distributions of the estimators, $\hat{C}_{ip}$, $\hat{C}_{ia}$ and $\hat{C}_{pp}$ are given only in Greenwich and Jahr-Schaffrath(1995) as follows.

Theorem 1 (Greenwich and Jahr-Schaffrath(1995))
If $\mu_4 = E(X - \mu)^4$ exists , then

$$(a)\quad \sqrt{n}(\hat{C}_{pp} - C_{pp}) \xrightarrow{d} \frac{1}{D^2}\{2(\mu - T)Z_1 + Z_2\} \equiv N(0, \sigma^2_{pp}),$$

$$\text{where } \sigma^2_{pp} = \frac{1}{D^4}\{(\mu_4 - \sigma^4) + 4(\mu - T)[\sigma^2(\mu - T) + \mu_3]\}.$$

$$(b)\quad \sqrt{n}(\hat{C}_{ia} - C_{ia}) \xrightarrow{d} N(0, \sigma^2_{ia}) \text{ , where } \sigma^2_{ia} = \frac{4(\mu - T)^2\sigma^2}{D^4}.$$

$$(c)\quad \sqrt{n}(\hat{C}_{ip} - C_{ip}) \xrightarrow{d} N(0, \sigma^2_{ip}) \text{ , where } \sigma^2_{ip} = \frac{(\mu_4 - \sigma^4)}{D^4}.$$

Proof See pp.65-68 of Greenwich and Jahr-Schaffrath(1995).

The two subindices, $\hat{C}_{ip}$ and $\hat{C}_{ia}$, are shown to be uncorrelated if the underlying distribution is normal(hence $\mu_3 = 0$). The asymptotic distribution of the estimator $\hat{C}_{pg}$ can be derived similarly. Also, the asymptotic distributions of two estimators for subindices can be derived in similar ways.

3. Bootstrapping PIIs

3.1 Bootstrap Algorithm

In this section, we introduce the bootstrap algorithm for deriving asymptotic distributions and confidence limits based on the bootstrap.

Let $X_1,\ X_2,\ \cdots,\ X_n$ be a random sample of size n from a possible population with distribution F, and let $t(X_1,\ X_2,\ \cdots,\ X_n; F)$ be the specified random variable of interest, and let F_n be the empirical distribution function of $X_1,\ X_2,\ \cdots,\ X_n$. Putting mass $1/n$ at each of the points $X_1,\ X_2,\ \cdots,\ X_n$, we get the bootstrap sample of size m, $X_1^*,\ X_2^*,\ \cdots,\ X_m^*$.

The bootstrap method is to approximate the distribution of $t(X_1, X_2, \cdots, X_n; F)$ under F by that of $t(X_1^*, X_2^*, \cdots, X_m^*; F)$. A formal description of the bootstrap algorithm goes as follows.

• Step 1 : Given $\chi_n = (X_1,\ X_2,\ \cdots,\ X_n)$, the bootstrap sample of size m $X_1^*, X_2^*, \cdots, X_n^*$ can be obtained with replacement, which is conditionally independent with common distribution F_n.

• Step 2 : From the bootstrap sample $X_1^*,\ X_2^*,\ \cdots,\ X_m^*$, compute the sample mean $\bar{X}^*$ and sample variance S^{*2}.

$$\bar{X}^* = \frac{1}{m}\sum_{i=1}^{m} X_i^*, \quad S^{*2} = \frac{1}{m}\sum_{i=1}^{m}(X_i^* - \bar{X}^*)^2.$$

• Step 3 : Compute the bootstrap plug-in estimatorof C_{pp},

$$\hat{C}_{pp}^* = \left\{ \frac{3\sqrt{S^{*2} + (\bar{X}^* - T)^2}}{min(USL - T,\ T - LSL)} \right\}^2.$$

3.2 Bootstrap Asymptotic Distributions

In the preceding section, we discussed the behavior of the asymptotic distributions of several PIIs. The bootstrap estimator for our index C_{pp} is obtained by replacing the bootstrap version with the estimator in (3).

In this section, we will derive its corresponding bootstrap asymptotic distribution, which may be useful unless process characteristic is normally distributed:

Theorem 2 If $\mu_4 = E(X-\mu)^4$ exists , then

$$\sqrt{m}(\hat{C}^*_{pp} - \hat{C}_{pp})|\mathcal{X}_n \xrightarrow{d} \frac{1}{D^2}\{2(\mu - T)Z_1 + Z_2\} \equiv N(0, \sigma^2_{pp}),$$
$$\text{where } \sigma^2_{pp} = \frac{1}{D^4}\{(\mu_4 - \sigma^4) + 4(\mu - T)[\sigma^2(\mu - T) + \mu_3]\}.$$

Proof As m and $n \to \infty$ we have

$$\begin{aligned}
&\sqrt{m}(\hat{C}^*_{pp} - \hat{C}_{pp})|\mathcal{X}_n \\
= \;& \sqrt{m}\left[\frac{\{S^{*2} + (\bar{X}^* - T)^2\}}{D^2} - \frac{\{S^2 + (\bar{X} - T)^2\}}{D^2}\right]\Big|\mathcal{X}_n \\
= \;& \frac{1}{D^4}\sqrt{m}\left[(S^{*2} - S^2) + (\bar{X}^* - \bar{X})(\bar{X}^* + \bar{X}) - 2T(\bar{X}^* - \bar{X})\right]\Big|\mathcal{X}_n \\
\xrightarrow{d}\; & \frac{1}{D^2}\{2(\mu - T)Z_1 + Z_2\},
\end{aligned}$$

by Lemma and Slutsky's Theorem. We also obtain

$$\sqrt{m}(\hat{C}^*_{ia} - \hat{C}_{ia})|\mathcal{X}_n \xrightarrow{d} N(0, \sigma^2_{ia}), \text{ where } \sigma^2_{ia} = \frac{4(\mu - T)^2\sigma^2}{D^4},$$
$$\sqrt{m}(\hat{C}^*_{ip} - \hat{C}_{ip})|\mathcal{X}_n \xrightarrow{d} N(0, \sigma^2_{ip}), \text{ where } \sigma^2_{ip} = \frac{(\mu_4 - \sigma^4)}{D^4}.$$

In the next subsections we will construct six bootstrap confidence limits for C_{pp} including Studentized Bootstrap(STUD), Hybrid Bootstrap(HYB), and Accelerated Bias-Corrected Bootstrap(ABC), which were theoretically summarized by Hall(1988).

3.3 Bootstrap Confidence Limits

For constructing the bootstrap confidence limits, consider the empirical distributions based on the bootstrap as below.

$$\hat{H}(x_\alpha) = P(\frac{\sqrt{m}(\hat{C}^*_{pp} - \hat{C}_{pp})}{S_{pp}} \le x_\alpha|\chi_n) = \alpha$$
$$\hat{K}(y_\alpha) = P(\frac{\sqrt{m}(\hat{C}^*_{pp} - \hat{C}_{pp})}{S^*_{pp}} \le y_\alpha|\chi_n) = \alpha, \tag{4}$$

where S_{pp}^2 implies the natural estimator of the variance σ_{pp}^2, that is

$$S_{pp}^2 = \frac{1}{D^4}\{(\hat{\mu}_4 - S^4) + 4(\bar{X} - T)[S^2(\bar{X} - T) + \hat{\mu}_3]\}$$
$$\hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3, \quad \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4$$

and S_{pp}^{*2} implies the bootstrap estimator of the variance σ_{pp}^2.

For each simulation study, a sample of size n was drawn and for each of size n, B bootstrap resamples were drawn from that single sample. This single simulation was then replicated N times. Thus, we are able to calculate the proportion of times the various bootstrap upper limits.

3.3.1 Standard Bootstrap(SB) Method

From the $B = 1000$ bootstrap estimates, $\hat{C}_{pp}^*(i)$, we calculate the sample average and the sample standard deviation as belows.

$$\bar{C}_{pp}^* = \frac{1}{1000}\sum_{i=1}^{1000}\hat{C}_{pp}^*(i), \quad se_B^* = \sqrt{\frac{1}{999}\sum_{i=1}^{1000}(\ \hat{C}_{pp}^*(i) - \bar{C}_{pp}^*\ )^2}$$

In fact, since the distribution of $\hat{C}_{pp}$ is approximately normal, we obtain the $(1-\alpha)100\%$ SB confidence limit for C_{pp}

$$(\ 0, \quad \hat{C}_{pp} + z_\alpha se_B^*\ ),$$

where z_α denotes α level quantile of the standard normal distribution.

3.3.2 Percentile Bootstrap(PB) Method

The PB method is used in more than half of the papers on bootstrap confidence intervals. But Hall(1988) pointed out a criticism of the PB method.

From the ordered collection of $\hat{C}_{pp}^*(i)$, $(\hat{C}_{pp}^*(1) \leq \cdots \leq \hat{C}_{pp}^*(B))$, we obtain the $(1-\alpha)100\%$ PB confidence interval for C_{pp}

$$(\ 0, \quad \hat{C}_{pp}^*((1-\alpha)B\ ).$$

3.3.3 Biased-Corrected Percentile Bootstrap(BCPB) Method

It is possible that bootstrap distribution for C_{pp} with small sample size may be biased. The BCPB method has been developed to correct for this potential bias.

First, using the ordered distribution of $\hat{C}^*_{pp}$, calculate the probability

$$p_0 = P(\ \hat{C}^*_{pp} \leq \hat{c}_{pp} | \chi_n\ ).$$

Second, calculate the quantile point z_0 and the probability P_U such that the following equations are satisfied.

$$z_0 = \phi^{-1}(p_0),\ P_U = \phi(2z_0 + z_{(1-\alpha)}).$$

where $\phi(\cdot)$ is the standard normal cumulative distribution function.
Finally, the $(1-\alpha)100\%$ BCPB confidence limit is

$$(\ 0,\quad \hat{C}^*_{pp}(P_U \cdot B)\ ).$$

3.3.4 Studentized Bootstrap(STUD) Method

Hall(1988) mentioned that the STUD method led to intervals which tended to be conservative in the sense that they had greater length and greater coverage than other competitors. So the STUD method does a better job than several other methods, provided that the variance estimate is chosen well. But this generalization can be failed in the case of distributions with exceptionally large positive kurtosis.

If process variance σ^2 is unknown, we can use the reasonable choice critical point, $\hat{y}_\alpha$ of the equation (4). The $(1-\alpha)100\%$ STUD confidence limit is

$$(\ 0,\quad \hat{C}_{pp} - \hat{y}_\alpha \frac{S_{pp}}{\sqrt{n}}\ ).$$

3.3.5 Hybrid Bootstrap(HYB) Method

The HYB method is used in almost all of the studies not using the PB method. Some users are even unaware that there is a difference between the HYB and PB method. Equal tailed intervals based on the HYB method and the PB method always have exactly the same length, but usually have different centers.

When process variance σ^2 is unknown, we can use the critical point, $\hat{x}_\alpha$ of the equation (4). The $(1-\alpha)100\%$ HYB confidence limit is

$$(\ 0,\quad \hat{C}_{pp} - \hat{x}_\alpha \frac{S_{pp}}{\sqrt{n}}\ ).$$

3.3.6 Accelerated Bias-Corrected Bootstrap(ABC) Method

It may occur that the bootstrap distributions obtained by using only a sample of the complete bootstrap distribution may be shifted higher or lower than would be expected. The ABC method is used to accelerate the BCPB method. That is, applied statisticians make frequent use of devices like transformations, bias corrections, and even acceleration adjustments, to improve the performance of the standard intervals. The ABC method enjoys useful properties of invariance under transformations, properties not shared by the STUD method, although the STUD method does a better job than the ABC method, provided the variance estimate is chosen correctly.

The acceleration constant, a, always measures the rate of change of standard error on a normalized scale. Consider $d = 2$ and the function

$$g(\mu) = g(\mu_1, \mu_2) = \frac{(\mu_1 - T)^2 + (\mu_2 - \mu_1^2)}{D^2}$$

The acceleration constant a for process indices defined by Hall(1988) as

$$a = \frac{1}{6\sqrt{n}\ \sigma_{pp}^3} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} a_i\ a_j\ a_k\ \mu_{ijk}$$

where $a_i = \partial g / \partial \mu_i$, $i = 1, 2$.

First, calculate the bootstrap estimator,$\hat{a}$, of the acceleration constant a for our incapability index C_{pp} as follows:

$$\hat{a} = \frac{1}{6\sqrt{m}\ S_{pp}^{*3}} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \hat{a}_i^*\ \hat{a}_j^*\ \hat{a}_k^*\ \hat{\mu}_{ijk}^*$$

We must keep in mind all estimators for acceleration constant a consist of bootstrap samples. This estimate, $\hat{a}$, approximately coincides with $\hat{a} \approx \frac{1}{6}\ Skew_{c=\hat{c}}(\dot{l}_c)$ where $\dot{l}_c$ is the score function, which is given by Efron(1987). This sounds different to compute, but it is in fact easier to get a good estimate of $\hat{a}$ than of z_0. For the process incapability index C_{pp}, consider the function $g(\mu_1, \mu_2) = \{(\mu_2 - \mu_1^2) + (\mu_1 - T)^2\}/D^2$. Then we can obtain $a_1 = -2T/D^2$, $a_2 = D^{-2}$.

Second, estimate $\hat{\beta}_{aU}$ which is $\hat{\beta}_{aU} = \phi(z_{(1-\alpha)} + 2z_0 + \hat{a}z_{(1-\alpha)}^2)$.

Then we get the 100% ABC confidence limit for the equation (4),

$$(\ 0,\quad \hat{C}_{pp}^* + \hat{x}_{\hat{\beta}_{aU}} \frac{S_{pp}}{\sqrt{n}}\ ).$$

Now, in the next section we will give simulated results for the PII C_{pp} to evaluate which upper confidence intervals are better than others with small sample. We restrict the original sample size n to the bootstrap sample size m.

< *Table* 1 > Four C_{pp} values used in the simulation study(M=T)

Distribution	μ	σ	USL	LSL	C_{pp}
Normal	13.5	0.625	16	10	0.64
Lognormal	13.5	0.870	16	10	1.00
$\chi^2(4)$	16.0	0.500	18	10	2.39
$t(4)$	16.0	0.660	18	10	2.50

4. Simulation Study

For each simulation study, a sample of size n was drawn and for each of size n, $B = 1000$ bootstrap resamples were drawn from that single sample. We assume that the bootstrap sample size m is equal to the original sample size n. This single simulation was then replicated $N = 1000$ times. Thus, we are able to calculate the proportion of times the various bootstrap upper limits. The frequency of coverage for the upper limit is a binomial random variable with $p = 0.95$ and $N = 1000$. Thus, a 99% confidence interval for the coverage proportion of a true 95% confidence limit is $0.95 \pm 2.576\sqrt{0.95 \cdot 0.05/1000} = (0.933, 0.967)$. Similarly,a 99% confidence interval for the coverage proportion of a true 90% confidence limit would be $0.90 \pm 0.0244 = (0.876, 0.924)$.

To compare the performance of the bootstrap upper confidence limits to the ones based on the assumption of a normal process, a series of simulations were undertaken. In this section we will discuss the behavior of six bootstrap confidence intervals for C_{pp}. We suppose four process distributions, normal, lognormal(skewed), chi-square(more skewed), and t(heavy-tailed) distributions. Last two distributions have four degrees of freedom. Four values of C_{pp} for each distribution are shown in < *Table* 1 >.

Greenwich and Jahr-Schaffrath(1995) presented the results of the simulation studies, which generally indicated that observed coverage rates tended to be lower than the preset nominal coverage level but the approximation accuracy seemed to be satisfactory for a sample size $n \geq 200$.

They also discussed the confidence intervals of these indices based on a small and a large sample($n \geq 80$). If a subgroup of size 5 is used for an $\bar{X}$-chart to monitor a process, 80 observations constitute 16 points on the chart. The number of observations, 80, is not prohibitively large for many processes. Therefore, for each distribution in < *Table* 1 >, the simulation was performed with original sample size $n = 30, 60, 80$.

However, their simulated coverage rates are indicated to be generally lower than the confidence level. To correct the accuracy, which is lower for a small sample, they used sample size denominator $\sqrt{n-k}$ with correction factor k, instead of $\sqrt{n}$.

Incidentally, the width of a confidence interval for a given significance level becomes wider as the sample size gets smaller.Their further simulation studies conducted for 95% two-sided confidence intervals indicate that the corrected denominator of $\sqrt{n-16}$ seems to work well for normal processes in general. This method is very simple but not proper to affirm for any other processes.

$< Table\ 2-1 >$ validates the performance of the 95% bootstrap upper confidence limits of the STUD method as being equivalent in the coverage performance for almost all indices under the assumption of an underlying normal process. With a large sample $n \geq 80$, the ABC limits work well too. When the process deteriorates, the coverages of all the methods except HYB tend to the value 0.95 with a large sample $n \geq 80$. In contrast to the PCIs(See Han et al.(2000)), these behaviors are same as for the lognormal process. The coverage speeds of SB and PB limits in the normal process are slower than those in the lognormal process. But the BCPB method is better in the lognormal process than in normal process with . For a heavy-tailed or more skewed process, all the methods resulted in proportions consistently below the expected value of 0.95. However, coverages between $\chi^2(4)$ and $t(4)$ make no noticeable difference, as shown by the results in Franklin and Wasserman (1991, 1992).

Greenwich and Jahr-Schaffrath(1995)'s simulation results indicated the observed coverage rates, i.e. the ratio of the number of confidence intervals that contain the true index to 10000(total number of intervals), tend to be lower than the preset significance level with a very large sample $n = 500$. Therefore, the result of $< Table\ 2-1 >$ shows the bootstrap method to be much more accurate for constructing the confidence interval of C_{pp} even with a sample $n \geq 80$.

From $< Table\ 2-2 >$, we see that almost all methods work well for a 90% bootstrap confidence interval with a large sample n=80 in the normal process. With samples of $n = 30,\ 60$, the STUD method only tends to be near the true value 90%. In contrast to PCIs(See Han et al.(1998)), the bias correction is not much more effective for the index C_{pp} with sample size $n = 30,\ 60$. For the lognormal process, when the process is highly capable, the STUD and SB methods are good. Moreover when the process is not capable, the BCPB and ABC methods additionally work well. In the heavy-tailed or more skewed process, as the coverages of 90% bootstrap confidence intervals are below the true value 90%. However, if the process is highly incapable, the cases of several methods have the proper coverages. Thus they are better than those of PCI in heavy-tailed or more skewed process.

$< Table\ 3 >$ records the average widths for each of the six bootstrap confidence intervals. Like the results of PCI, the average width decreased as n increased for all intervals and all indices. The shortest average width of the 90% bootstrap confidence intervals for C_{pp} is obtained from the PB and HYB methods, not from the ABC method. However, the differences among the methods are not noticeable with $n = 80$. The average widths of the PII, C_{pp} are larger than those of PCIs. The more the process deteriorates the larger the average widths, so those of a more skewed, $\chi^2(4)$ process are larger than those of a heavy-tailed, $t(4)$, process.

< *Table* 4 > contains the standard deviations of the widths for the six methods. The standard deviations of the widths of the PII, C_{pp} are larger than those of PCIs. The results indicated the similar patterns of the PCIs in the normal and lognormal processes. In contrast to the average widths, the STUD method has larger standard deviations in the $t(4)$ process than in the $\chi^2(4)$ process.

5. Conclusion

In this paper we have derived bootstrap asymptotic distribution for the PII C_{pp}, and constructed six bootstrap confidence limits for C_{pp}. We also have provided some simulation results under normal, lognormal, chi-square and t-distributions to consider skewness as well as kurtosis.

Our experiment results show that the coverage proportions for the STUD, ABC, BCPB bootstrap control limits based on normality are generally stable and worked well. But PB method is the worst. In particular, ABC method need much more computing time and memory than the others. For non-normal processes , STUD, HYB, and ABC bootstrap methods are generally stable and worked well.

Even though most confidence intervals do not work perfectly well so that the coverage percentages were quite lower, they reveal that our bootstrap confidence limits perform significantly than those on the asymptotic normality.

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< *Table* 2 − 1 >Coverage percentages for the 95% BUCL of C_{pp}

		Normal			Lognormal		
	n	30	60	90	30	60	90
	SB	0.888*	0.893*	0.927*	0.885*	0.910*	0.929*
$\mu = 13.5$	PB	0.893*	0.890*	0.933	0.894*	0.904*	0.916*
$\sigma = 0.625$	BCPB	0.914*	0.913*	0.929*	0.899*	0.926*	0.934
$C_{pp} = 0.64$	STUD	0.954	0.933	0.951	0.944	0.945	0.950
	HYB	0.882*	0.891*	0.917*	0.867*	0.904*	0.914*
	ABC	0.925*	0.920*	0.934	0.905*	0.934	0.941
	SB	0.891*	0.888*	0.926*	0.872*	0.913*	0.922*
$\mu = 13.5$	PB	0.898*	0.880*	0.932*	0.882*	0.906*	0.913*
$\sigma = 0.67$	BCPB	0.912*	0.913*	0.932*	0.891*	0.920*	0.930*
$C_{pp} = 1$	STUD	0.954	0.935	0.956	0.934	0.945	0.947
	HYB	0.880*	0.886*	0.917*	0.858*	0.897*	0.907*
	ABC	0.926*	0.918*	0.942	0.900*	0.931*	0.937
	SB	0.922*	0.920*	0.949	0.926*	0.937	0.924*
$\mu = 13.5$	PB	0.921*	0.916*	0.946	0.926*	0.919*	0.916*
$\sigma = 0.5$	BCPB	0.932*	0.919*	0.937	0.936	0.931*	0.934
$C_{pp} = 0.5$	STUD	0.947	0.934	0.943	0.943	0.945	0.939
	HYB	0.924*	0.913*	0.931*	0.927*	0.926*	0.925*
	ABC	0.935	0.925*	0.941	0.939	0.937	0.937*
	SB	0.920*	0.917*	0.946	0.919*	0.933	0.924*
$\mu = 13.5$	PB	0.919*	0.914*	0.942	0.924*	0.919*	0.921*
$\sigma = 0.66$	BCPB	0.931*	0.920*	0.935	0.933	0.928*	0.934
$C_{pp} = 2.5$	STUD	0.950	0.940	0.945	0.945	0.943	0.940
	HYB	0.917*	0.908*	0.929*	0.914*	0.923*	0.923*
	ABC	0.933	0.925*	0.941	0.936	0.930*	0.941
		$\chi^2(4)$			$t(4)$		
	SB	0.810*	0.847*	0.860*	0.798*	0.843*	0.813*
$\mu = 13.5$	PB	0.822*	0.850*	0.859*	0.810*	0.841*	0.817*
$\sigma = 0.625$	BCPB	0.840*	0.868*	0.879*	0.828*	0.861*	0.839*
$C_{pp} = 0.64$	STUD	0.913*	0.914*	0.917*	0.879*	0.897*	0.886*
	HYB	0.792*	0.824*	0.839*	0.779*	0.825*	0.796*
	ABC	0.856*	0.885*	0.897*	0.849*	0.877*	0.853*
	SB	0.792*	0.829*	0.853*	0.770*	0.824*	0.798*
$\mu = 13.5$	PB	0.808*	0.839*	0.851*	0.781*	0.825*	0.800*
$\sigma = 0.67$	BCPB	0.827*	0.863*	0.871*	0.804*	0.840*	0.821*
$C_{pp} = 1.00$	STUD	0.906*	0.912*	0.917*	0.869*	0.893*	0.873*
	HYB	0.777*	0.810*	0.823*	0.752*	0.806*	0.782*
	ABC	0.843*	0.884*	0.890*	0.829*	0.862*	0.839*
	SB	0.890*	0.897*	0.902*	0.911*	0.928*	0.909*
$\mu = 13.5$	PB	0.893*	0.883*	0.900*	0.909*	0.926*	0.905*
$\sigma = 0.5$	BCPB	0.897*	0.902*	0.910*	0.913*	0.926*	0.911
$C_{pp} = 0.5$	STUD	0.928*	0.930*	0.926*	0.933	0.938	0.916*
	HYB	0.870*	0.881*	0.895*	0.903*	0.921*	0.897*
	ABC	0.909*	0.913*	0.921*	0.914*	0.927*	0.916*
	SB	0.880*	0.888*	0.900*	0.896*	0.919*	0.903*
$\mu = 13.5$	PB	0.883*	0.880*	0.893*	0.895*	0.918*	0.895*
$\sigma = 0.66$	BCPB	0.894*	0.897*	0.906*	0.905*	0.922*	0.904*
$C_{pp} = 2.5$	STUD	0.923*	0.923*	0.926*	0.929*	0.931*	0.913*
	HYB	0.864*	0.878*	0.890*	0.894*	0.910*	0.886*
	ABC	0.903*	0.906*	0.917*	0.909*	0.924*	0.911*

*Indicates a proportion significantly different from the expected value of 0.95 (at $\alpha = 0.01$)

< *Table* 2 − 2 > Coverage percentages for the 90% BUCL of C_{pp}

		Normal			Lognormal		
	n	30	60	90	30	60	90
	SB	0.850*	0.862*	0.893	0.856*	0.878	0.892
$\mu = 13.5$	PB	0.848*	0.843*	0.895	0.857*	0.861*	0.860*
$\sigma = 0.625$	BCPB	0.855*	0.859*	0.879	0.855*	0.868*	0.872*
$C_{pp} = 0.64$	STUD	0.888	0.876	0.893	0.895	0.889	0.880
	HYB	0.850*	0.844*	0.878	0.843*	0.867*	0.870*
	ABC	0.842*	0.852*	0.872*	0.944*	0.867*	0.866*
	SB	0.856*	0.856*	0.890	0.847*	0.884	0.887
$\mu = 13.5$	PB	0.852*	0.840*	0.894	0.848*	0.862*	0.855*
$\sigma = 0.67$	BCPB	0.851*	0.858*	0.878	0.846*	0.865*	0.873*
$C_{pp} = 1$	STUD	0.890	0.875*	0.891	0.888	0.882	0.883
	HYB	0.849*	0.843*	0.876	0.834*	0.855*	0.869*
	ABC	0.846*	0.847*	0.877	0.839*	0.856*	0.862*
	SB	0.870*	0.876	0.908	0.876	0.896	0.872*
$\mu = 13.5$	PB	0.866*	0.855*	0.894	0.873*	0.870*	0.857*
$\sigma = 0.5$	BCPB	0.874*	0.854*	0.879	0.878	0.874*	0.879
$C_{pp} = 0.5$	STUD	0.890	0.872*	0.883	0.883	0.893	0.877
	HYB	0.871*	0.855*	0.872*	0.872*	0.878	0.875*
	ABC	0.874*	0.858*	0.880	0.874*	0.880	0.878
	SB	0.869*	0.876	0.904	0.872*	0.898	0.877
$\mu = 13.5$	PB	0.863*	0.856*	0.891	0.873*	0.875*	0.867*
$\sigma = 0.66$	BCPB	0.873*	0.859*	0.877	0.874*	0.876	0.879
$C_{pp} = 2.5$	STUD	0.895	0.878	0.886	0.885	0.890	0.881
	HYB	0.868*	0.851*	0.871*	0.863*	0.879	0.875*
	ABC	0.872*	0.861*	0.881	0.870*	0.875*	0.879
		$\chi^2(4)$			$t(4)$		
	SB	0.799*	0.831*	0.841*	0.787*	0.832*	0.803*
$\mu = 13.5$	PB	0.802*	0.821*	0.826*	0.791*	0.814*	0.789*
$\sigma = 0.625$	BCPB	0.803*	0.820*	0.825*	0.794*	0.822*	0.791*
$C_{pp} = 0.64$	STUD	0.870*	0.860*	0.862*	0.837*	0.847*	0.829*
	HYB	0.784*	0.798*	0.809*	0.771*	0.806*	0.777*
	ABC	0.798*	0.815*	0.826*	0.790*	0.812*	0.786*
	SB	0.778*	0.815*	0.839*	0.760*	0.813*	0.789*
$\mu = 13.5$	PB	0.789*	0.808*	0.817*	0.763*	0.802*	0.800*
$\sigma = 0.67$	BCPB	0.790*	0.815*	0.823*	0.769*	0.799*	0.821*
$C_{pp} = 1$	STUD	0.862*	0.861*	0.860*	0.831*	0.847*	0.873*
	HYB	0.768*	0.787*	0.798*	0.745*	0.787*	0.782*
	ABC	0.782*	0.813*	0.818*	0.772*	0.794*	0.839*
	SB	0.874*	0.876	0.874*	0.875*	0.901	0.873*
$\mu = 13.5$	PB	0.869*	0.839*	0.857*	0.871*	0.887	0.860*
$\sigma = 0.5$	BCPB	0.863*	0.846*	0.855*	0.862*	0.877	0.845*
$C_{pp} = 0.5$	STUD	0.893	0.871*	0.868*	0.881	0.892	0.854*
	HYB	0.855*	0.848*	0.856*	0.868*	0.892	0.850*
	ABC	0.860*	0.842*	0.856*	0.849*	0.868*	0.838*
	SB	0.865*	0.868*	0.873*	0.869*	0.900*	0.873*
$\mu = 13.5$	PB	0.860*	0.840*	0.850*	0.858*	0.885	0.853*
$\sigma = 0.66$	BCPB	0.862*	0.845*	0.855*	0.856*	0.878	0.839*
$C_{pp} = 2.5$	STUD	0.887	0.866*	0.867*	0.880	0.885	0.849*
	HYB	0.852*	0.844*	0.854*	0.866*	0.881	0.842*
	ABC	0.854*	0.834*	0.851*	0.844*	0.866*	0.835*

*Indicates a proportion significantly different from the expected value of 0.90 (at $\alpha = 0.01$)

< *Table* 3 > Average of the widths of the 90% BCI of C_{pp}

		Normal			Lognormal		
	n	30	60	90	30	60	90
	SB	0.484	0.345	0.303	0.508	0.367	0.319
$\mu = 13.5$	PB	0.480	0.343	0.301	0.504	0.365	0.318
$\sigma = 0.625$	BCPB	0.490	0.346	0.303	0.515	0.368	0.320
$C_{pp} = 0.64$	STUD	0.580	0.379	0.325	0.628	0.411	0.348
	HYB	0.480	0.343	0.301	0.504	0.365	0.318
	ABC	0.496	0.348	0.304	0.523	0.371	0.322
	SB	0.790	0.563	0.494	0.843	0.611	0.533
$\mu = 13.5$	PB	0.784	0.560	0.491	0.835	0.607	0.529
$\sigma = 0.67$	BCPB	0.801	0.565	0.495	0.856	0.613	0.533
$C_{pp} = 1.00$	STUD	0.968	0.626	0.536	1.089	0.699	0.589
	HYB	0.784	0.560	0.491	0.835	0.607	0.529
	ABC	0.812	0.569	0.497	0.873	0.619	0.538
	SB	0.667	0.475	0.414	0.673	0.484	0.420
$\mu = 13.5$	PB	0.664	0.473	0.413	0.671	0.482	0.419
$\sigma = 0.5$	BCPB	0.669	0.475	0.413	0.675	0.484	0.420
$C_{pp} = 0.5$	STUD	0.711	0.490	0.424	0.721	0.501	0.431
	HYB	0.664	0.473	0.413	0.671	0.482	0.419
	ABC	0.670	0.475	0.413	0.677	0.484	0.419
	SB	0.889	0.633	0.552	0.904	0.651	0.565
$\mu = 13.5$	PB	0.885	0.631	0.551	0.900	0.648	0.563
$\sigma = 0.66$	BCPB	0.892	0.633	0.550	0.907	0.650	0.564
$C_{pp} = 2.5$	STUD	0.955	0.656	0.567	0.980	0.677	0.582
	HYB	0.885	0.631	0.551	0.900	0.648	0.563
	ABC	0.894	0.633	0.551	0.911	0.651	0.565
		$\chi^2(4)$			$t(4)$		
	SB	0.722	0.540	0.474	0.629	0.515	0.416
$\mu = 13.5$	PB	0.712	0.534	0.469	0.615	0.499	0.409
$\sigma = 0.625$	BCPB	0.738	0.546	0.479	0.638	0.522	0.420
$C_{pp} = 0.64$	STUD	1.164	0.712	0.597	1.281	1.165	0.614
	HYB	0.712	0.534	0.469	0.615	0.499	0.409
	ABC	0.764	0.557	0.488	0.667	0.539	0.431
	SB	1.207	0.909	0.798	1.097	0.914	0.739
$\mu = 13.5$	PB	1.187	0.896	0.789	1.070	0.881	0.720
$\sigma = 0.67$	BCPB	1.238	0.920	0.809	1.114	0.928	0.758
$C_{pp} = 1.00$	STUD	2.163	0.268	1.054	2.540	2.384	1.311
	HYB	1.187	0.896	0.789	1.070	0.881	0.720
	ABC	1.288	0.943	0.826	1.171	0.961	0.778
	SB	0.762	0.553	0.482	0.667	0.49	0.421
$\mu = 13.5$	PB	0.756	0.550	0.479	0.662	0.49	0.419
$\sigma = 0.5$	BCPB	0.768	0.555	0.484	0.670	0.49	0.421
$C_{pp} = 0.5$	STUD	0.899	0.605	0.520	0.777	0.56	0.453
	HYB	0.756	0.550	0.479	0.662	0.49	0.419
	ABC	0.777	0.558	0.486	0.677	0.50	0.423
	SB	1.056	0.769	0.671	0.905	0.680	0.573
$\mu = 13.5$	PB	1.046	0.764	0.659	0.897	0.671	0.570
$\sigma = 0.66$	BCPB	1.067	0.771	0.667	0.910	0.681	0.574
$C_{pp} = 2.5$	STUD	1.278	0.856	0.727	1.102	0.821	0.634
	HYB	1.046	0.764	0.659	0.897	0.671	0.570
	ABC	1.082	0.778	0.671	0.924	0.689	0.578

< *Table* 4 > Standard deviation of the widths of the 90% BCI of C_{pp}

		Normal			Lognormal		
	n	30	60	90	30	60	90
$\mu = 13.5$, $\sigma = 0.625$, $C_{pp} = 0.64$	SB	0.141	0.073	0.053	0.161	0.086	0.066
	PB	0.140	0.073	0.053	0.159	0.085	0.066
	BCPB	0.146	0.074	0.053	0.165	0.086	0.066
	STUD	0.218	0.090	0.064	0.260	0.118	0.080
	HYB	0.140	0.073	0.053	0.159	0.085	0.066
	ABC	0.150	0.074	0.054	0.172	0.088	0.067
$\mu = 13.5$, $\sigma = 0.67$, $C_{pp} = 1.00$	SB	0.248	0.126	0.093	0.300	0.161	0.124
	PB	0.244	0.125	0.092	0.294	0.159	0.123
	BCPB	0.256	0.127	0.093	0.307	0.161	0.122
	STUD	0.398	0.161	0.115	0.531	0.237	0.159
	HYB	0.244	0.125	0.092	0.294	0.159	0.123
	ABC	0.265	0.130	0.095	0.323	0.167	0.126
$\mu = 13.5$, $\sigma = 0.5$, $C_{pp} = 0.5$	SB	0.105	0.054	0.040	0.110	0.056	0.044
	PB	0.104	0.054	0.041	0.110	0.057	0.045
	BCPB	0.106	0.057	0.044	0.112	0.060	0.049
	STUD	0.119	0.056	0.042	0.126	0.060	0.045
	HYB	0.104	0.054	0.041	0.110	0.057	0.045
	ABC	0.291	0.108	0.057	0.113	0.060	0.047
$\mu = 13.5$, $\sigma = 0.66$, $C_{pp} = 2.5$	SB	0.151	0.077	0.057	0.164	0.084	0.066
	PB	0.150	0.077	0.058	0.163	0.085	0.067
	BCPB	0.153	0.081	0.060	0.166	0.088	0.069
	STUD	0.178	0.082	0.060	0.196	0.093	0.068
	HYB	0.150	0.077	0.058	0.163	0.085	0.067
	ABC	0.156	0.081	0.061	0.168	0.088	0.069
		$\chi^2(4)$			$t(4)$		
$\mu = 13.5$, $\sigma = 0.625$, $C_{pp} = 0.64$	SB	0.367	0.224	0.183	0.642	0.718	0.294
	PB	0.357	0.217	0.180	0.592	0.618	0.283
	BCPB	0.382	0.237	0.183	0.622	0.748	0.278
	STUD	0.992	0.472	0.353	4.317	7.475	0.890
	HYB	0.357	0.217	0.180	0.592	0.618	0.283
	ABC	0.410	0.246	0.192	0.731	0.814	0.297
$\mu = 13.5$, $\sigma = 0.67$, $C_{pp} = 1.00$	SB	0.666	0.411	0.334	1.217	1.382	0.566
	PB	0.647	0.397	0.329	1.123	1.185	0.534
	BCPB	0.694	0.434	0.335	1.178	1.441	0.529
	STUD	2.128	0.957	0.713	9.279	1.115	1.783
	HYB	0.647	0.397	0.329	1.123	1.185	0.534
	ABC	0.749	0.454	0.351	1.385	1.566	0.567
$\mu = 13.5$, $\sigma = 0.5$, $C_{pp} = 0.5$	SB	0.225	0.128	0.101	0.294	0.247	0.135
	PB	0.221	0.126	0.100	0.278	0.226	0.133
	BCPB	0.233	0.132	0.102	0.293	0.247	0.129
	STUD	0.351	0.175	0.131	0.756	0.701	0.215
	HYB	0.221	0.126	0.100	0.278	0.226	0.133
	ABC	0.241	0.136	0.104	0.325	0.277	0.135
$\mu = 13.5$, $\sigma = 0.66$, $C_{pp} = 2.5$	SB	0.340	0.196	0.156	0.465	0.425	0.216
	PB	0.334	0.193	0.154	0.436	0.383	0.211
	BCPB	0.353	0.202	0.156	0.461	0.429	0.204
	STUD	0.563	0.282	0.212	1.379	1.476	0.386
	HYB	0.334	0.193	0.154	0.436	0.383	0.211
	ABC	0.368	0.210	0.161	0.521	0.479	0.215