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Recurrence Relation and Characterization of The Rayleigh Distribution Using Order Statistics [1]

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Abstract

In this paper the single and product moments of order statistics of the doubly truncated Rayleigh distribution are studied. Some recurrence relations of order statistics are derived. Using order statistics, also characterization of the Rayleigh distribution are derived.

Key Words and Phrases: Rayleigh distribution, Recurrence Relation, Characterization.

1. Introduction

Order statistics and their moments have assumed considerable interest in recent years and the moments of order statistics have been tabulated quite extensively for several distributions, for example see Arnold, Balakrishnan and Nagaraja (1992). Also Khan, Parvez and Yaqub (1983) and Balakrishnan and Malik (1985) etc have studied several recurrence relation for the some distribution. That is, Balakrishnan, Malik and Ahmed(1988) and Malik, Balakrishnan and Ahmed (1988) reviewed several recurrence relations and identities for single and product moments of order statistics for specific distributions. Mohie El-Din, Mahmoud and Abu-Youssef(1991) derived the single and product moments of order statistics from doubly truncated parabolic and skewed distribution. Rita and Balakrishnan(1996) derived recurrence relations for single and product moments of order statistics of truncated exponential distribution.

On the other hand Lin(1988) and Mohie El-Din, Mahmoud and Abu-Youssef(1991) have obtained the Characterization same distribution by moment of order statistics

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Now we consider a p.d.f. $f(x)$ of doubly truncated Rayleigh distribution. Then

$$f(x) = \frac{(2/\theta^2)xe^{-(x/\theta)^2}}{P-Q}, \qquad Q_1 \le x \le P_1, \qquad \theta > 0, \tag{1.1}$$

where P_1 and Q_1 are arbitrary constants and

$$P = 1 - e^{-(P_1/\theta)^2} \quad and \quad Q = 1 - e^{-(Q_1/\theta)^2}.$$

Letting

$$Q_2 = \frac{1-Q}{P-Q}, \quad \text{and} \quad P_2 = \frac{1-P}{P-Q}. \tag{1.2}$$

Then the cumulative distribution function is given by $F(x) = Q_2 - \frac{\theta^2}{2x} f(x)$. That is, $1 - F(x) = -P_2 + \frac{e^{-(x/\theta)^2}}{P-Q}$. Hence we have the relation that

$$f(x) = P_2\frac{2x}{\theta^2} + \frac{2x}{\theta^2}(1 - F(x)) = Q_2\frac{2x}{\theta^2} - \frac{2x}{\theta^2}F(x).$$

Also we have easily obtain that the kth moment

$$\mu^{(k)} = \frac{(2/\theta^2)}{(k+2)}[P_2P_1^{(k+2)} - Q_2Q_1^{(k+2)} + \mu^{(k+2)}],$$

where P_2 and Q_2 are given in (1.2). From (1.1) we also have the mean of the distribution to be

$$\mu^{(1)} = Q_1Q_2 - P_1P_2 + \theta\sqrt{\pi}[\Phi(\frac{\sqrt{2}P_1}{\theta}) - \Phi(\frac{\sqrt{2}Q_1}{\theta})], \tag{1.3}$$

where $\Phi(\cdot)$ is a standard normal c.d.f.

In this paper, we study for the single and product moments of the order statistics for the doubly truncated Rayleigh distribution. Also characterization of the Rayleigh distribution are derived.

2. Moments of the Order Statistics and Relations for Single Moments

Let $x_{r:n} \le x_{r+1:n}, \quad r = 1, 2, ..., n-1,$ be the order statistics from random sample of size n from the doubly truncated Rayleigh distribution (1.1), then

$$f_{r:n}(x) = C_{r:n}F(x)^{(r-1)}(1 - F(x))^{(n-r)}f(x). \tag{2.1}$$

Expanding $(1-F(x))^{(n-r)}$ binomially in powers of $F(x)$, we get

$$f_{r:n}(x) = C_{r:n}\sum_{j=0}^{n-r}\sum_{i=0}^{n-j-1} a_{n-r,j}a_{n-j-1,i}Q_2^i(\frac{\theta^2}{2x})^{(l-1)}f(x)^l,$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$, $\quad a_{\alpha,\beta} = (-1)^{(\alpha-\beta)}\binom{\alpha}{\beta}$ and $n-j-i=k$.
From (1.1) and (2.1), we can obtain the following moment $\mu_{r:n}$.

$$\begin{aligned}\mu_{r:n} &= \int_{Q_1}^{P_1} xf_{r:n}(x)dx \\ &= C_{r:n}\sum_{j=0}^{n-r}\sum_{i=0}^{n-j-1} a_{n-r,j}\; a_{n-j-1,i}\; Q_2^i\;\frac{1}{l}\;[Q_1Q_2^l - P_1P_2^l \\ &\quad + \sqrt{\frac{\pi}{l}}\,\theta\,\frac{1}{(P-Q)^k}\,\{\Phi(P_1\sqrt{2l}/\theta) - \Phi(Q_1\sqrt{2l}/\theta)\}].\end{aligned}$$

Theorem 2.1 For $n \geq 1$ and $Q_1 \leq x_{r:n} \leq P_1$

$$\mu_{1:n}^{(k)} = \frac{2n}{(k+2)\theta^2}[\mu_{1:n}^{(k+2)} + P_2\mu_{1:n-1}^{(k+2)} - Q_1^{(k+2)}Q_2], \tag{2.2}$$

and for $2 \leq r \leq n$

$$\mu_{r:n}^{(k)} = \frac{2n}{(k+2)\theta^2}[P_2\mu_{r:n-1}^{(k+2)} - P_2\mu_{r-1:n-1}^{(k+2)}] + \frac{2(n-r+1)}{(k+2)\theta^2}[\mu_{r:n}^{(k+2)} - \mu_{r-1:n}^{(k+2)}]. \tag{2.3}$$

Proof By (2.1)

$$\begin{aligned}\mu_{1:n}^{(k)} &= \int_{Q_1}^{P_1} C_{1:n}\;x^k\;(1-F(x))^{(n-1)}f(x)dx \\ &= \frac{2nP_2}{(k+2)\theta^2}[-Q_1^{(k+2)} + \mu_{1:n-1}^{(k+2)}] + \frac{2n}{(k+2)\theta^2}[-Q_1^{(k+2)} + \mu_{1:n}^{(k+2)}].\end{aligned}$$

Simplifying the resulting expression, we have the relation (2.2). For $2 \leq r \leq n$

$$\begin{aligned}\mu_{r:n}^{(k)} &= \int_{Q_1}^{P_1} C_{r:n}x^rF(x)^{(r-1)}(1-F(x))^{(n-r)}f(x)dx \\ &= C_{r:n}\frac{2P_2}{\theta^2}\frac{n}{(k+2)C_{r:n}}[-\mu_{r-1:n-1}^{(k+2)} + \mu_{r:n-1}^{(k+2)}] + C_{r:n}\frac{2}{\theta^2}\frac{(n-r-1)}{(k+2)C_{r:n}} \\ &\quad [-\mu_{r-1:n}^{(k+2)} + \mu_{r:n}^{(k+2)}].\end{aligned}$$

Hence, we obtain the relation in (2.3).

3. Product Moments of Order Statistics and Recurrence Relation for Product Moments

It is well known that the joint density function of $x_{r:n}$ and $x_{s:n}(1 \le r \le s \le n)$ is given by

$$f_{r,s:n}(x,y) = C_{r,s:n}F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)}f(x)f(y), \tag{3.1}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. Also the Equation (3.1) may be take the following form

$$\begin{aligned} f_{r,s:n}(x,y) =& C_{r,s:n}(Q_2 - \frac{1}{P-Q}\, e^{-(x/\theta)^2})^{(r-1)}(-\frac{1}{P-Q}\, e^{-(y/\theta)^2} \\ &+ \frac{1}{P-Q}\, e^{-(x/\theta)^2})^{(s-r-1)}(-P_2 + \frac{1}{P-Q}\, e^{-(y/\theta)^2})^{(n-s)}. \end{aligned}$$

By binomial expanding we have

$$\begin{aligned} f_{r,s:n}(x,y) =& \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{m=0}^{n-s}(-1)^{(s-i-j-2+m)}\binom{r-1}{i}\binom{s-r-1}{j}\binom{n-s}{m}Q_2^i\, P_2^m \\ &\times (\frac{1}{P-Q}\, e^{-(x/\theta)^2})^{(r-1-i+j)}(\frac{1}{P-Q}\, e^{-(y/\theta)^2})^{(n-r-m-j-1)}f(x)f(y). \end{aligned}$$

Let $l_1 = r+j-i,\quad l_2 = n-m-r-j,\quad l_3 = l_1+l_2 = n-m-i$ and $\Lambda(i,j,m) = (-1)^{(s-i-j-2+m)}\binom{r-1}{i}\binom{s-r-1}{j}\binom{n-s}{m}\, Q_2^i\, P_2^m$. Then

$$f_{r,s:n}(x,y) = \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{m=0}^{n-s}\Lambda(i,j,m)(\frac{1}{P-Q})^{l_3}e^{-(x/\theta)^2(l_1-1)}\, e^{-(y/\theta)^2(l_2-1)}f(x)f(y). \tag{3.2}$$

Hence we can obtain the product moments. That is,

$$\begin{aligned} \mu_{r,s:n} &= \int_{Q_1}^{P_1}\int_{x}^{P_1} xyf_{r,s:n}(x,y)dxdy \\ &= C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{m=0}^{n-s}\Lambda(i,j,m)(\frac{1}{P-Q})^{l_3}D(x,y), \end{aligned}$$

where

$$\begin{aligned} D(x,y) &= \int_{Q_1}^{P_1}\int_{x}^{P_1} xye^{-(x/\theta)^2(l_1-1)}e^{-(y/\theta)^2(l_2-1)}f(x)f(y)dydx \\ &= \frac{(2/\theta^2)^2}{(P-Q)^2}\int_{Q_1}^{P_1} x^2e^{-l_1(x/\theta)^2}\frac{\theta^2}{2l_1}[xe^{-l_1(x/\theta)^2} - P_1e^{-l_2(P_1/\theta)^2} + G_1(x)]dx. \end{aligned} \tag{3.3}$$

In (3.3), define $G_1(x) = \int_x^{P_1} e^{-l_2(y/\theta)^2}dy$. Then, by trasformation, we get the following result

$$G_1(x) = \sqrt{\frac{\pi}{l_2}}\theta[\Phi(\sqrt{2l_2}P_1/\theta) - \Phi(\sqrt{2l_2}Q_1/\theta)],$$

where the $\Phi(\cdot)$ is a c.d.f. of standard normal variate. Hence, by several calculation, we have

$$\begin{aligned}\mu_{r,s:n} =&C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{m=0}^{n-s}\Lambda(i,j,m)(\frac{1}{P-Q})^{l_3}\frac{(2/\theta^2)}{l_2}\\ &\times[H_1 - P_1(1-P)_2^l H_2 + \sqrt{\frac{\pi}{l_2}}\theta\{\Phi(\sqrt{2l_2}P_1/\theta)H_2 - H_3\}], \qquad (3.4)\end{aligned}$$

where

$$\begin{aligned}H_1 &= \frac{\theta^2}{2l_3}[Q_1^2(1-Q)^{l_3} + P_1^2(1-P)^{l_3} + \frac{\theta^2}{l_3}\{(1-Q)^{l_3} - (1-P)^{l_3}\}],\\ H_2 &= \frac{\theta^2}{2l_1}[Q_1(1-Q)^{l_1} - P_1(1-P)^{l_1} + \sqrt{\frac{\pi}{l_1}}\theta\{\Phi(\sqrt{2l_2}P_1/\theta) - \Phi(\sqrt{2l_2}Q_1/\theta)\}],\\ H_3 &= \frac{\theta^2}{2l_1}[Q_1(1-Q)^{l_1}\Phi(\sqrt{2l_2}Q_1/\theta) - P_1(1-P)^{l_1}\Phi(\sqrt{2l_2}P_1/\theta)\\ &+ I_1 - \sqrt{\frac{l_2}{\pi}}\frac{1}{\theta}\{\frac{\theta^2}{2l_3}(1-Q)^{l_3} - (1-P)^{l_3}\}]. \qquad (3.5)\end{aligned}$$

In (3.5)

$$\begin{aligned}I_1 =&\sqrt{\frac{\pi}{l_1}}\theta[\frac{1}{2}\{\Phi(\sqrt{2l_1}P_1/\theta) - \Phi(\sqrt{2l_1}Q_1/\theta)\} + V(\sqrt{2l_1}P_1/\theta,\ \sqrt{\frac{l_2}{l_1}}\sqrt{2l_1}P_1/\theta)\\ &- V(\sqrt{2l_1}Q_1/\theta,\ \sqrt{\frac{l_2}{l_1}}\sqrt{2l_1}Q_1/\theta)], \qquad (3.6)\end{aligned}$$

and $V(h,k) = \frac{1}{2\pi}\int_0^h\int_0^{\frac{kx}{h}} e^{-\frac{1}{2}(x^2+y^2)}dxdy$. Therefore

$$\begin{aligned}H_3 =&\frac{\theta^2}{2l_1}[Q_1(1-Q)^{l_1}\Phi(\sqrt{2l_2}Q_1/\theta) - P_1(1-P)^{l_1}\Phi(\sqrt{2l_2}P_1/\theta)\\ &+\{\sqrt{\frac{\pi}{l_1}}\theta\{\frac{1}{2}(\Phi(\sqrt{2l_1}P_1/\theta) - \Phi(\sqrt{2l_1}Q_1/\theta)) + V(\sqrt{2l_1}P_1/\theta,\ \sqrt{2l_2}P_1/\theta)\\ &- V(\sqrt{2l_1}Q_1/\theta,\ \sqrt{2l_2}Q_1/\theta)\}\} - \{\sqrt{\frac{l_2}{\pi}}\frac{\theta}{2l_3}((1-Q)^{l_3} - (1-P)^{l_3})\}].\end{aligned}$$

Hence we have

$$\mu_{r,s:n} = C_{r,s:n} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \sum_{m=0}^{n-s} \Lambda(i,j,m) \frac{1}{l_2} \times [K_1 - P_1 P_2^{l_2} \frac{1}{l_1} K_2 + \sqrt{\frac{\pi}{l_2}} \theta \{ \Phi(\sqrt{2l_2} P_1/\theta) (\frac{1}{P-Q})^{l_2} K_2 - K_3 \}], \quad (3.7)$$

where

$$K_1 = \frac{1}{l_3}[Q_1^2 Q_2^{l_3} + P_1^2 P_2^{l_3} + \frac{\theta^2}{l_3}(Q_2^{l_3} - P_2^{l_3})]$$

$$K_2 = Q_1 Q_2^{l_1} - P_1 P_2^{l_1} \sqrt{\frac{\pi}{l_1}} \theta (\frac{1}{P-Q})^{l_1} [\Phi(\sqrt{2l_2} P_1/\theta) - \Phi(\sqrt{2l_2} Q_1/\theta)]$$

$$K_3 = \frac{1}{l_1} \Big[(\frac{1}{P-Q})^{l_2} \{ Q_1 Q_2^{l_1} \Phi(\sqrt{2l_2} Q_1/\theta) - P_1 P_2^{l_1} \Phi(\sqrt{2l_2} P_1/\theta) \} + (\frac{1}{P-Q})^{l_3} \times \{ \sqrt{\frac{\pi}{l_1}} \theta (\frac{1}{2} (\Phi(\sqrt{2l_2} P_1/\theta) - \Phi(\sqrt{2l_2} Q_1/\theta)) + V(\sqrt{2l_1} P_1/\theta, \sqrt{2l_2} P_1/\theta) - V(\sqrt{2l_1} Q_1/\theta, \sqrt{2l_2} Q_1/\theta) \} - \sqrt{\frac{l_2}{\pi}} \frac{\theta}{2l_3} (Q_2^{l_3} - P_2^{l_3}) \Big].$$

The quantities V(h,k) of (3.6) have been very extensively tabulated by Nicholson(1943), and Yamauti(1972).

Theorem 3.1 For $1 \leq r \leq n-1$ and $i, j > 0$

$$\mu_{r,r+1:n}^{(i,j)} = \frac{2nP_2}{(j+2)\theta^2} [\mu_{r,r+1:n-1}^{(i,j+2)} - \mu_{r:n-1}^{(i,j+2)}] + \frac{2(n-r)}{(j+2)\theta^2} [\mu_{r,r+1:n}^{(i,j+2)} - \mu_{r:n}^{(i,j+2)}], \quad (3.8)$$

and for $1 \leq r < s \leq n$, $n-r \geq 2$ and $0 \leq j$

$$\mu_{r,s:n}^{(i,j)} = \frac{2nP_2}{(j+2)\theta^2} [\mu_{r,s:n-1}^{(i,j+2)} - \mu_{r,s-1:n-1}^{(i,j+2)}] + \frac{2(n-s+1)}{(j+2)\theta^2} [\mu_{r,s:n}^{(i,j+2)} - \mu_{r,s-1:n}^{(i,j+2)}]. \quad (3.9)$$

Proof By (3.1)

$$\mu_{r,r+1:n}^{(i,j)} = \int_{Q_1}^{P_1} \int_{x}^{P_1} C_{r,r+1:n} x^i y^j F(x)^{(r-1)} (1-F(y))^{(n-r-1)} f(x) f(y) dx dy$$

$$= C_{r,r+1:n} \int_{Q_1}^{P_1} x^i F(x)^{(r-1)} f(x) M_1(x) dx,$$

where

$$M_1(x) = \int_x^{P_1} y^j (1-F(y))^{(n-r-1)} f(y)dy$$
$$= \frac{2P_2}{(j+2)\theta^2}\Big[-x^{(j+2)}(1-F(x))^{(n-r-1)} + (n-r-1)\int_x^{P_1} y^{(j+2)}$$
$$(1-F(y))^{(n-r-2)} f(y)dy\Big] + \frac{2}{(j+2)\theta^2}\Big[-x^{(j+2)}(1-F(x))^{(n-r)}$$
$$+(n-r)\int_x^{P_1} y^{(j+2)}(1-F(y))^{(n-r-1)} f(y)dy\Big].$$

Therefore (3.8).

On the other hand,

$$\mu_{r,s:n}^{(i,j)} = \int_{Q_1}^{P_1}\int_{Q_1}^{y} C_{r,s:n} x^i y^j F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)} f(x)f(y)dxdy$$
$$= C_{r,s:n}\int_{Q_1}^{P_1} x^i F(x)^{(r-1)} f(x) M_2(x)dx,$$

where

$$M_2(x) = \int_x^{P_1} y^j (F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)} f(y)dy$$
$$= \frac{2P_2}{(j+2)\theta^2}\Big[-(s-r-1)\int_x^{P_1} y^{(j+2)}(F(y)-F(x))^{(s-r-2)}(1-F(y))^{(n-s)} f(y)dy$$
$$+(n-s)\int_x^{P_1} y^{(j+2)}(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s-1)} f(y)dy\Big]$$
$$+\frac{2}{(j+2)\theta^2}\Big[-(s-r-1)\int_x^{P_1} y^{(j+2)}(F(y)-F(x))^{(s-r-2)}(1-F(y))^{(n-s+1)}$$
$$\times f(y)dy + (n-s+1)\int_x^{P_1} y^{(j+2)}(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)} f(y)dy\Big].$$

We have conclusion (3.9).

Theorem 3.2 For $n \geq 2$ and $i,\ j \geq 0$

$$\mu_{1,2:n}^{(i,j)} = \frac{2}{(i+2)\theta^2}\big[nQ_2\mu_{1:n-1}^{(i+j+2)} - \mu_{2:n}^{(i+j+2)} + \mu_{1,2:n}^{(i+2,j)}\big] - Q_2 Q_1^{(i+2)}\mu_{1:n-1}^{(j)}, \tag{3.10}$$

and for $2 \leq r \leq n-1$ and $i, j > 0$

$$\mu_{r,r+1:n}^{(i,j)} = \frac{2nQ_2}{(i+2)\theta^2}\big[\mu_{r:n-r}^{(i+j+2)} - \mu_{r-1,r:n-r}^{(i+2,j)}\big]$$
$$+\frac{2r}{(i+2)\theta^2}\big[\mu_{r,r+1:n}^{(i+2,j)} - \mu_{r+1:n}^{(i+j+2)}\big]. \tag{3.11}$$

Proof First we consider, $\mu_{1,2:n}^{(i,j)}$. Then

$$\mu_{1,2:n}^{(i,j)} = \int_{Q_1}^{P_1}\int_{Q_1}^{y} C_{1,2:n}x^i y^j (1-F(y))^{(n-2)} f(x)f(y)dxdy.$$

Integration about x and spliting the integral accordingly into four, we get

$$\begin{aligned}\mu_{1,2:n}^{(i,j)} =& n(n-1)\frac{2Q_2}{(i+2)\theta^2}\Big[\int_{Q_1}^{P_1} y^{(i+j+2)}(1-F(y))^{(n-2)}f(y)dy \\ &- Q_1^{(i+2)}\int_{Q_1}^{P_1} y^j(1-F(y))^{(n-2)}f(y)dy\Big] \\ &- n(n-1)\frac{2}{(i+2)\theta^2}\Big[\int_{Q_1}^{P_1} y^{(i+j+2)}F(y)(1-F(y))^{(n-2)}f(y)dy \\ &- \int_{Q_1}^{P_1}\int_{Q_1}^{y} x^{(i+2)}y^j(1-F(x))^{(n-2)}f(x)f(y)dxdy\Big] \\ =& \frac{2}{(i+2)\theta^2}\big[nQ_2\mu_{1:n-1}^{(i+j+2)} - \mu_{2:n}^{(i+j+2)} + \mu_{1,2:n}^{(i+2,j)}\big] - Q_2Q_1^{(i+2)}\mu_{1:n-1}^{(j)}.\end{aligned}$$

Therefore we obtain the Equation (3.10).

Next we consider $\mu_{r,r+1:n}^{(i,j)}$. Then

$$\begin{aligned}\mu_{r,r+1:n}^{(i,j)} &= \int_{Q_1}^{P_1}\int_{Q_1}^{y} C_{r,r+1:n}x^i y^j F(x)^{(r-1)}(1-F(y))^{(n-r-1)}f(x)f(y)dxdy \\ &= C_{r,r+1:n}\int_{Q_1}^{P_1} y^j(1-F(y))^{(n-r-1)}f(y)M_3(y)dy,\end{aligned}$$

where

$$\begin{aligned}M_3(y) =& \int_{Q_1}^{y} x^i F(x)^{(r-1)}f(x)dx \\ =& \frac{2Q_2}{(i+2)\theta^2}\big[y^{(i+2)}F(y)^{(r-1)} - (r-1)\int_{Q_1}^{y} x^{(i+2)}F(x)^{(r-2)}f(x)dx\big] \\ &- \frac{2}{(i+2)\theta^2}\big[y^{(i+2)}F(y)^r - r\int_{Q_1}^{y} x^{(i+2)}F(x)^{(r-1)}f(x)dx\big].\end{aligned}$$

We obtain the equation (3.11).

Theorem 3.3 For $1 \le r < s \le n, \quad s-r > 2 \quad$ and $\quad i,\ j \ge 0$,

$$\mu_{r,s:n}^{(i+2,j)} = \mu_{r+1,s:n}^{(i+2,j)} + \frac{(i+2)\theta^2}{2r}\mu_{r,s:n}^{(i,j)} + \frac{nQ_2}{r}\big[\mu_{r-1,s-1:n-1}^{(i+2,j)} - \mu_{r,s-1:n-1}^{(i+2,j)}\big]. \tag{3.12}$$

Proof By (3.1)

$$\mu_{r,s:n}^{(i,j)} = \int_{Q_1}^{P_1}\int_{Q_1}^{y} C_{r,s:n} x^i y^j F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)} f(x)f(y)dxdy$$
$$= C_{r,s:n}\int_{Q_1}^{P_1} y^j(1-F(y))^{(n-s)} f(y) M_4(y)dy,$$

where

$$M_4(y) = \int_{Q_1}^{y} x^i F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)} f(x)dx$$
$$= \frac{2Q_2}{(i+2)\theta^2}\Big[-(r-1)\int_{Q_1}^{y} x^{(i+2)}F(x)^{(r-2)}(F(y)-F(x))^{(s-r-1)}f(x)dx$$
$$+(s-r-1)\int_{Q_1}^{y} x^{(i+2)}F(x)^{(r-1)}(F(y)-F(x))^{(s-r-2)}f(x)dx\Big]$$
$$+\frac{2}{(i+2)\theta^2}\Big[r\int_{Q_1}^{y} x^{(i+2)}F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)}f(x)dx$$
$$-(s-r-1)\int_{Q_1}^{y} x^{(i+2)}F(x)^{r}(F(y)-F(x))^{(s-r-2)}f(x)dx\Big].$$

Therefore, we can obtain

$$\mu_{r,s:n}^{(i,j)} = \frac{2nQ_2}{(i+2)\theta^2}\left[\mu_{r,s-1:n-1}^{(i+2,j)} - \mu_{r-1,s-1:n-1}^{(i+2,j)}\right] + \frac{2r}{(i+2)\theta^2}\left[\mu_{r,s:n}^{(i+2,j)} - \mu_{r+1,s:n}^{(i+2,j)}\right].$$

Hence we have

$$\mu_{r,s:n}^{(i+2,j)} = \mu_{r+1,s:n}^{(i+2,j)} + \frac{(i+2)\theta^2}{2r}\mu_{r,s:n}^{(i,j)} + \frac{nQ_2}{r}\left[\mu_{r-1,s-1:n-1}^{(i+2,j)} - \mu_{r,s-1:n-1}^{(i+2,j)}\right].$$

4. Characterization of the Rayleigh Distribution

We well known that the unconditional p.d.f. of $(s-r)th$ order statistics in a sample of size (n-r) is given by, $x_{r:n} = x \le x_{s:n} = y$, [see David(1981)]

$$f(x_{s:n}|x_{r:n}=x) = \frac{(n-r)!(F(y)-F(x))^{(s-r-1)}(1-F(y))^{(n-s)}f(y)}{(n-s)!(s-r-1)!(1-F(x))^{(n-r)}}, \tag{4.1}$$

where $f(x_{s:n}|x_{r:n}=x)$ is the conditional density of $x_{s:n}$ given $x_{r:n}=x$ and the sample drawn from $\frac{f(y)}{1-F(x)}$, $x \le y$, which is obtained from the truncated parent distribution on the left truncated at x. Therefore, from the left truncated at x, it can be seen that

$$Q_1 = x,\ P_1 = \infty,\ P = 1,\ Q = F(x),\ P_2 = 0,\ \text{and}\ Q_2 = 1. \tag{4.2}$$

Then (4.1) can be written as

$$f(x_{r+1:n}|x_{r:n} = x) = \frac{(n-r)!(1-F(y))^{(n-r-1)}f(y)}{(1-F(x))^{(n-r)}}. \tag{4.3}$$

Similarly, if the parent distribution truncated from the right at y, ($x \le y$ and $r < s$), then

$$f(x_{r:n}|x_{s:n} = y) = \frac{(s-1)!F(x)^{(r-1)}(F(y)-F(x))^{(s-r-1)}f(x)}{(r-1)!(s-r-1)!F(y)^{(s-1)}}. \tag{4.4}$$

In the case of the right truncated at y, we have

$$Q_1 = 0,\ P_1 = y,\ P = F(y),\ Q = 0,\ P_2 = \frac{1-F(y)}{F(y)},\quad \text{and}\quad Q_2 = \frac{1}{F(y)}. \tag{4.5}$$

Hence (4.4) takes the form

$$f(x_{1:n}|x_{2:n} = y) = \frac{f(y)}{F(x)},\quad x \le y.$$

Theorem 4.1 If $F(x) < 1$, $(0 < x < \infty)$ be a c.d.f of the random variable X,

$$E(X^k) < \infty \quad \text{and}\ F(0) = 0,\ \text{then}\quad F(x) = 1 - e^{-(x/\theta)^2} \qquad x > 0,\ \theta > 0.$$

if and only if

$$\frac{2}{\theta^2}\mu_{1:n-r}^{(k+2)} - \frac{(k+2)}{n-r}\mu_{1:n-r}^{(k)} = \frac{2}{\theta^2}x^{(k+2)} \tag{4.6}$$

for the left truncated at x.

Proof By (2.3) and (4.2), we have

$$\frac{(k+2)}{(n-r)}\mu_{1:n-r}^{(k)} = \frac{2}{\theta^2}\mu_{1:n-r}^{(k+2)} - \frac{2}{\theta^2}x^{(k+2)}.$$

Hence we can obtain the Equation(4.6).

Assume that equation (4.6) hold. Then

$$[(n-r)\frac{2}{\theta^2}\int_x^\infty y^{(k+2)}(1-F(y))^{(n-r-1)}f(y)dy$$
$$-(k+2)\int_x^\infty y^k(1-F(y))^{(n-r-1)}f(y)dy] = \frac{2}{\theta^2}x^{(k+2)}(1-F(x))^{(n-r)}.$$

Differentiating both sides with respect to x, then

$$[-\frac{2}{\theta^2}(n-r)x^{(k+2)}(1-F(x))^{(n-r-1)}f(x)+(k+2)x^k(1-F(x))^{(n-r-1)}f(x)]$$
$$=[\frac{2}{\theta^2}(k+2)x^{(k+1)}(1-F(x))^{(n-r)}\frac{2}{\theta^2}x^{(k+2)}\times(1-F(x))^{(n-r-1)}(n-r)f(x)].$$

Therefore

$$[x^k(1-F(x))^{(n-r-1)}f(x)\{-\frac{2}{\theta^2}(n-r)x^2+(k+2)\}]$$
$$=[\frac{2}{\theta^2}x^k(1-F(x))^{(n-r-1)}\{(k+2)x(1-F(x))-(n-r)x^2f(x)\}].$$

Hence

$$f(x)[-\frac{2}{\theta^2}(n-r)x^2+(k+2)]=\frac{2}{\theta^2}[(k+2)x(1-F(x))-(n-r)x^2f(x)].$$

That is,

$$f(x)(k+2)=\frac{2}{\theta^2}(k+2)x(1-F(x)).$$

Hence we have $\frac{f(x)}{1-F(x)}=\frac{2x}{\theta^2}$.

Theorem 4.2 If $F(x)<1, 0<x<\infty$ be a c.d.f of the random variable X,

$$E(X^k)<\infty \quad \text{and } F(0)=0, \text{ then} F(x)=1-e^{-(x/\theta)^2} \qquad x>0,\ \theta>0.$$

if and only if

$$(k+2)\mu_{1:1}^{(k)}-\frac{2}{\theta^2}\mu_{1:1}^{(k+2)}=\frac{2}{\theta^2}x^{(k+2)}\frac{1-F(x)}{F(x)} \tag{4.7}$$

for the right truncated at x.

Proof From (2.3) and the assumption $\mu_{n:n-1}^{(k)}=P_1^{(k)},\quad n\geq 1$, we have easily the relation (4.7).

Let (4.7) hold. Differentiating both sides with respect to x, by (2.3) and (4.5), we have

$$[(k+2)x^kf(x)-\frac{2}{\theta^2}x^{(k+2)}f(x)]=[\frac{2}{\theta^2}\{(k+2)x^{(k+1)}(1-F(x))-x^{(k+2)}f(x)\}].$$

Therefore

$$[x^kf(x)\{(k+2)-\frac{2}{\theta^2}x^2\}]=[\frac{2}{\theta^2}x^k\{(k+2)x(1-F(x))-x^2f(x)\}].$$

Hence

$$f(x)[(k+2)-\frac{2}{\theta^2}x^2f(x)]=\frac{2}{\theta^2}[(k+2)x(1-F(x))-\frac{2}{\theta^2}x^2f(x)].$$

That is, $\frac{f(x)}{1-F(x)}=\frac{2}{\theta^2}x$.

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