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Variational-Type Inequalities on Reflexive Banach Spaces

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In this paper, we consider the existence of solutions to the variational-type inequalities for single-valued mappings and set-valued mappings on reflexive Banach spaces using Fan's section theorem.

1. Introduction and preliminaries

Variational inequalities introduced by Hartman and Stampacchia[5] have been extended and generalized in various directions as a powerful tool of current mathematical technology.

Recently, Behera and Panda[3] introduced variational-type inequalities for single-valued mappings.

In this paper, we consider the existence of the solutions to the variational-type inequalities for single-valued mappings on reflexive Banach spaces, under different conditions from Behera and Panda[3]. And we consider the existence of the solutions to the variational-type inequalities for set-valued mappings on reflexive Banach spaces.

Now we introduce the following famous Fan's section theorem[4].

Theorem 1. 1. *Let* K *be a nonempty compact convex subset of a Hausdorff topological vector space* X. *Let* A *be a subset of* $K \times K$ *satisfying the following conditions ;*

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(1) *for each* $x \in K$, $(x,x) \in A$,

(2) *for each fixed* $x \in K$, *the set* $A_x := \{y \in K : (x,y) \in A\}$ *is closed in* K,

(3) *for each fixed* $y \in K$, *the set* $A^y := \{x \in K : (x,y) \notin A\}$ *is convex in* K.

Then there exists an $x_0 \in K$ *such that* $K \times \{x_0\} \subset A$.

Throughout this paper, we denote by $\langle y, x \rangle$ the duality mapping between elements $y \in X^*$ and $x \in X$.

2. In case of single-valued mappings

Now we consider variational-type inequalities for single-valued mappings.

Theorem 2. 1. *Let* K *be a nonempty closed convex and bounded subset of a reflexive Banach space* X *and* X^* *be the dual of* X. *Assume that* $T : K \to X^*$, $\theta : K \times K \to X$ *and* $\eta : K \times K \to \mathbb{R}$ *are mappings satisfying the following conditions ;*

(1) $\langle T(x), \theta(x,x) \rangle + \eta(x,x) = 0$ *for each* $x \in K$,

(2) *the mapping*

$$x \mapsto \langle T(y), \theta(x,y) \rangle + \eta(y,x)$$

of K *into* $\mathbb{R}$ *is convex for each* $y \in K$,

(3) *the mapping*

$$y \mapsto \langle T(y), \theta(x,y) \rangle + \eta(y,x)$$

of K *into* $\mathbb{R}$ *is continuous for each* $x \in K$.

Then there exists an $x_0 \in K$ *such that for all* $y \in K$

$$\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0.$$

Proof. Let $A := \{(x,y) \in K \times K : \langle T(y), \theta(x,y) \rangle + \eta(y,x) \geq 0\}$, then it is easily shown that $(x,x) \in A$. For each fixed $x \in K$,

$$\begin{aligned} A_x &:= \{y \in K : (x,y) \in A\} \\ &= \{y \in K : \langle T(y), \theta(x,y) \rangle + \eta(y,x) \geq 0\} \end{aligned}$$

is closed. Indeed, let $\{y_\lambda\}$ be a net in A_x such that $y_\lambda \to y_0$. Since $y_\lambda \in A_x$, we have

$$\langle T(y_\lambda), \theta(x, y_\lambda)\rangle + \eta(y_\lambda, x) \geq 0.$$

Hence by the condition (3),

$$\langle T(y_\lambda), \theta(x, y_\lambda)\rangle + \eta(y_\lambda, x) \rightarrow \langle T(y_0), \theta(x, y_0)\rangle + \eta(y_0, x).$$

Thus

$$\langle T(y_0), \theta(x, y_0)\rangle + \eta(y_0, x) \geq 0.$$

Hence $y_0 \in A_x$ and A_x is closed.

On the other hand, for each fixed $y \in K$,

$$\begin{aligned} A^y := &\{x \in K: (x, y) \notin A\} \\ = &\{x \in K: \langle T(y), \theta(x, y)\rangle + \eta(y, x) < 0\} \end{aligned}$$

is convex. In fact, let $x_1, x_2 \in A^y$, $\alpha \in [0,1]$ and $z = \alpha x_1 + (1-\alpha)x_2$, then by the condition (2),

$$\begin{aligned} &\langle T(y), \theta(z, y)\rangle + \eta(y, z) \\ = &\langle T(y), \theta(\alpha x_1 + (1-\alpha)x_2, y)\rangle + \eta(y, \alpha x_1 + (1-\alpha)x_2) \\ \leq &\alpha[\langle T(y), \theta(x_1, y)\rangle + \eta(y, x_1)] + (1-\alpha)[\langle T(y), \theta(x_2, y)\rangle + \eta(y, x_2)] \\ < &0, \end{aligned}$$

hence $z \in A^y$ and A^y is convex. Thus by Theorem 1. 1, there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$. This implies that there exists an $x_0 \in K$ such that

$$\langle T(x_0), \theta(y, x_0)\rangle + \eta(x_0, y) \geq 0,$$

for all $y \in K$.

Remark 2. 2. We obtained the same result under different conditions in [3].

3. In case of set-valued mappings

Definition 3. 1[2]. Let X, Y be two topological vector spaces and $T: X \rightarrow 2^Y$ be a set-valued mapping. T is said to be upper semicontinuous(briefly, u.s.c.) at $x_0 \in X$ if for any open neighbourhood N containing $T(x_0)$ there exists a neighbourhood M of x_0 such that $T(M) \subset N$. T is said to be u.s.c. if T is u.s.c. at every point $x \in X$.

Definition 3. 2[6]. Let X, Y be two topological vector spaces and $T: X \to 2^Y$ be a set-valued mapping. T is said to be closed at $x \in X$ if for each nets $\{x_\lambda\}$ converging to x and $\{y_\lambda\}$ converging to y such that $y_\lambda \in T(x_\lambda)$ for all λ, we have $y \in T(x)$. T is said to be closed if it is closed at every point $x \in X$.

Lemma 3. 1[1]. *Let X, Y be two topological vector spaces and $T: X \to 2^Y$ be a set-valued mapping.*

(1) *if K is a compact subset of X, and T is u.s.c. and compact-valued, then $T(K)$ is compact.*

(2) *if T is u.s.c. and compact-valued, then T is closed.*

Now we consider variational-type inequalities for set-valued mappings.

Theorem 3. 2. *Let K be a nonempty closed convex and bounded subset of a reflexive Banach space X and X^* be the dual of X. Assume that $T: K \to 2^{X^*}$ is an u.s.c. mapping with compact-values, $\theta: K \times K \to X$ is a bounded mapping and $\eta: K \times K \to \mathbb{R}$ is a mapping satisfying the following conditions ;*

(1) *for each $x \in K$, there exists $t \in T(x)$ such that $\langle t, \theta(x,x) \rangle + \eta(x,x) = 0$,*

(2) *a mapping*

$$x \mapsto \langle t, \theta(x,y) \rangle + \eta(y,x)$$

of K into $\mathbb{R}$ is convex for all $y \in K$ and for all $t \in T(y)$,

(3) *for each $x \in K$, mappings $y \mapsto \theta(x,y)$ and $y \mapsto \eta(y,x)$ are continuous.*

Then there exists an $x_0 \in K$ and $t_0 \in T(x_0)$ such that for any $y \in K$

$$\langle t_0, \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0.$$

Proof. Let $A := \{(x,y) \in K \times K:$ there exists $t \in T(y)$ such that $\langle t, \theta(x,y) \rangle + \eta(y,x) \geq 0\}$, then it is easily shown that $(x,x) \in A$. For each fixed $x \in K$,

$$A_x := \{y \in K: (x,y) \in A\}$$
$$= \{y \in K: \text{there exists } t \in T(y) \text{ such that } \langle t, \theta(x,y) \rangle + \eta(y,x) \geq 0\}$$

is closed. Indeed, let $\{y_\lambda\}$ be a net in A_x such that $y_\lambda \to y_0$. Since $y_\lambda \in A_x$, there exists $t_\lambda \in T(y_\lambda)$ such that $\langle t_\lambda, \theta(x, y_\lambda) \rangle + \eta(y_\lambda, x) \geq 0$.

Since K is weakly compact, by Lemma 3. 1(1), $T(K)$ is compact and hence without loss of generality, we can assume that there exists $t_0 \in T(K)$ such that $t_\lambda \to t_0$. By Lemma 3. 1(2), T is closed, hence $t_0 \in T(y_0)$. By the condition (3), we have

$$
\begin{aligned}
& | \langle t_\lambda, \theta(x, y_\lambda)\rangle + \eta(y_\lambda, x) - (\langle t_0, \theta(x, y_0)\rangle + \eta(y_0, x))| \\
\le & | \langle t_\lambda, \theta(x, y_\lambda)\rangle - \langle t_0, \theta(x, y_0)\rangle | + | \eta(y_\lambda, x) - \eta(y_0, x) | \\
\le & | \langle t_\lambda - t_0, \theta(x, y_\lambda)\rangle | + | \langle t_0, \theta(x, y_\lambda) - \theta(x, y_0)\rangle | + | \eta(y_\lambda, x) - \eta(y_0, x) | \\
\le & ||t_\lambda - t_0|| \, ||\theta(x, y_\lambda)|| + ||t_0|| \, ||\theta(x, y_\lambda) - \theta(x, y_0)|| + | \eta(y_\lambda, x) - \eta(y_0, x) |
\end{aligned}
$$

$\to 0$ as $\lambda \to \infty$. Consequently,

there exists $t_0 \in T(y_0)$ such that $\langle t_0, \theta(x, y_0)\rangle + \eta(y_0, x) \ge 0$.

Hence $y_0 \in A_x$ and A_x is closed.

On the other hand, for each fixed $y \in K$,

$$
\begin{aligned}
A^y := & \{x \in K: (x, y) \notin A\} \\
= & \{x \in K: \text{ for all } t \in T(y), \ \langle t, \theta(x, y)\rangle + \eta(y, x) < 0\}
\end{aligned}
$$

is convex. In fact, let $x_1, x_2 \in A^y$, $\alpha \in [0, 1]$ and $z = \alpha x_1 + (1-\alpha)x_2$, then for all $t \in T(y)$,

$$
\begin{aligned}
& \langle t, \theta(z, y)\rangle + \eta(y, z) \\
= & \langle t, \theta(\alpha x_1 + (1-\alpha)x_2, y)\rangle + \eta(y, \alpha x_1 + (1-\alpha)x_2) \\
\le & \alpha[\langle t, \theta(x_1, y)\rangle + \eta(y, x_1)] + (1-\alpha)[\langle t, \theta(x_2, y)\rangle + \eta(y, x_2)] \\
< & 0,
\end{aligned}
$$

hence $z \in A^y$. By Theorem 1. 1, there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$. This implies that there exists an $x_0 \in K$ and $t_0 \in T(x_0)$ such that for all $y \in K$, $\langle t_0, \theta(y, x_0)\rangle + \eta(x_0, y) \ge 0$.

Corollary 3. 3. *Considering $T: K \to X^*$ in Theorem 3. 2, we obtain Theorem 2. 1 as a corollary.*

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