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◉ 論 文

A Study on the Selective Use of Higher Order Elements

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고차 요소의 선택적 사용에 대한 연구

김 진 환*

Key Words : Mixed Degree Finite Elements(혼합 차수 유한 요소), Hierarchical Basis Function(계층 기저 함수), Degree of Freedom(자유도), C0-continuity(C0 연속성), Subparametric Element(저 자유도 요소), Poisson Equation(포아손방정식), Stokes Equation(스톡스 방정식)

초 록

일차원 및 이차원의 단순한 문제에 대하여 계층 요소를 사용한 혼합 차수 유한 요소해의 정확성 및 수렴성을 조사하였다. 이러한 작업은 임의의 차수를 가진 블록들을 조합하여 요소를 구성함으로서 이루어질 수 있다. 블록간의 연결성이 유지될 수 있는 블록의 구성과 요소 생성에 대하여는 코드 개발과 관련하여 설명하고 있으며, 서로 다른 차수를 가진 인접 블록간의 해의 연속성에 대하여는 계층 요소의 구성과 관련하여 서술되었다. 수치적 결과는 블록의 차수를 잘 선택함으로서 유한 요소해의 수렴성과 정확성을 증가시킬 수 있음을 보여주고 있으며, 고차 요소 영역을 너무 많이 할당하여 선형 요소의 영역이 너무 적을 경우에는 경계 조건에 따라 오차가 내부로 전파됨을 보여준다. 또한 세분화된 요소에 대한 고차 보간의 경우, 해의 수렴성이 저해될 수 있음이 발견되었다.

1. Introduction

The p-version, in finite element method, is known to converge faster than the h-version. However, the implement by the standard Lagrangian type elements is not easy due to the complication of element construction.

One way of overcoming this difficulty is to use the Hierarchical elements, which require, say, only 9 nodes geometrically in two dimensions. This means that the element geometry should not be curved much to meet at most the quadratic interpolation. However, the solution degrees can go higher than the quadratic, which is referred to as a subparametric element. The efficiency of hierarchical basis functions has

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been cited by Zienkiewicz at el[1] and Robinson[2].

In order to generate elements easily, the considered domain is divided into several subdomains which are called as a "block". The block with an arbitrary solution degree should have enough information to make globally consistent numbering for elements and nodes. Then the solution of an element is constructed according to the specified degree of the block which contains the element. Also the elements at the block interface should be given a special attention to meet the continuity requirement. For higher order solutions, elements should have the required number of nodal points corresponding to the solution degrees of freedom. But this way might lose the optimal generation of elements due to the necessity of too many nodal points within an element. But using the hierarchical shape functions, the elemental shape can be restricted to the quadratic Lagrangian quadrature or cubical element, and the remaining degrees of freedom can lie on the edge nodes, the face nodes or the interior nodes. This geometrical restriction can simplify the construction of a higher order element, especially at the block interfaces. The term "higher order element" means quadratic or cubic element.

This study is to investigate qualities of solutions obtained from different regional degrees. For this, two problems are expected; the first one is to find a way of connecting topologically and geometrically quadrature and/or cubical blocks of various shapes. the second one is to find a way of solution continuity between elements adjacent to the interfaces when the solution degrees of blocks are arbitrary.

Since the solution of elements are interpolated by the hierarchical shape functions, the maximum number of nodes needed in an element becomes 9(27) in two(three) dimensions. The present code makes use of this fact explicitly in generating a nodal variables and can simplify code management.

A brief explanation of the code, written by Visual Basic language[4], together with a systematic way of constructing an elemental solution is presented in Ch. 2. In Ch. 3, some numerical examples in one and two dimensional cases are considered.

2. Construction of Blocks and Elemental Solutions

A two dimensional quadrature block consists of 4 vertices and 4 edges, and a three dimensional cubical block consists of 8 verteces, 12 edges and 6 faces. Depending on the curvature of a block, a user can decide whether to use a 2nd degree interpolation or a 3rd. One may use the serendipity shape function to make the input nodal points minimum.

To define a block and hence to generate elemental solutions, the present code incorporates Type variables to define Blocks, Edges, Elements and Nodes. The type variables for Blocks, Elements and Nodes consist of necessary informations defining each type variables. The choice of Edge variable is accrued from the reasoning that for a structured grid, the edge connection seems more basic than the face connection. A detailed description can be found in Ref. (5).

The input data for blocks are shown in Table 1, and the shape of blocks and the elements generated are shown in Fig. 1 (a) and (b) respectively.

Elements of Hierarchical shape functions are same as Lagrangian element in the linear case, but they differ in the higher order cases. In quadratic elements, the geometries are same as

Table 1 Input data example for linear blocks

Block #		1	2	3	4
Type		Cube	Cube	Cube	Quad
Block Deg		1	1	1	1
Npoints		8	8	8	4
Elem. Deg.	Geom	1	1	1	1
	Soln	1	1	1	1
# of Elem.	Dir. 1	4	4	2	2
	Dir. 2	6	3	5	5
	Dir. 3	4	3	3	1
Prog. Ratio	Dir. 1	1.2	1.0	1.0	1.5
	Dir. 2	0.8	0.5	1.0	0.5
	Dir. 3	1.0	1.8	1.0	1.0

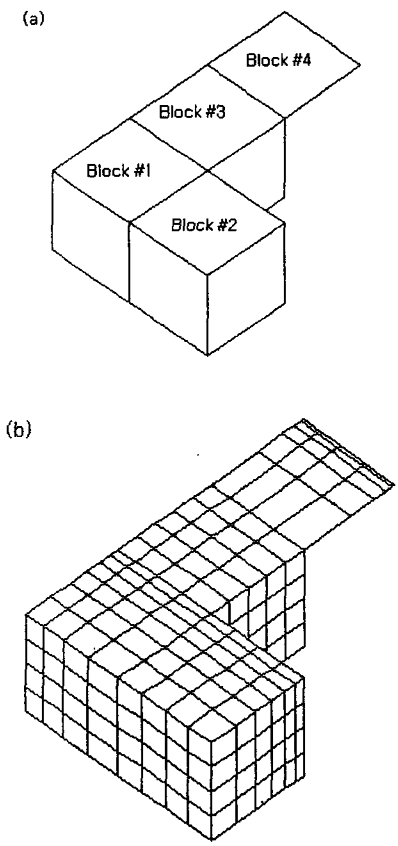

Fig. 1 An Example of block data in table 1 (a) Blocks (b) Elements

freedom at the edge nodes are not the nodal values but the 2nd order tangential derivatives. that for the Lagrangian case but the degrees of For the N^{th} order elements, the element degrees of freedom are obtained by adding some necessary dofs to the $(N-1)^{th}$ order element. In two dimensional case, one dof is added to each edge node and the remainings to the interior node. Followings are hierarchical shape functions in one dimension[1]:

$$h_1 = 0.5(1. - \eta) : h_2 = 0.5(1. + \eta) \qquad h_3 = 0.5(\eta^2 - 1.) \tag{2.1}$$

Followings are the hierarchical shape functions by tensor product of Eq.(2.1) for two dimensional quadratic element:

$$\begin{aligned}
\phi_1 &= h_1(\eta_1)\ h_1(\eta_2)\\
\phi_2 &= h_2(\eta_1)\ h_1(\eta_2)\\
\phi_3 &= h_2(\eta_1)\ h_2(\eta_2)\\
\phi_4 &= h_1(\eta_1)\ h_2(\eta_2)\\
\phi_5 &= -h_3(\eta_1)\ h_1(\eta_2)\\
\phi_6 &= -h_2(\eta_1)\ h_3(\eta_2)\\
\phi_7 &= -h_3(\eta_1)\ h_2(\eta_2)\\
\phi_8 &= -h_1(\eta_1)\ h_3(\eta_2)\\
\phi_9 &= h_3(\eta_1)\ h_3(\eta_2)
\end{aligned} \tag{2.2}$$

The degrees of freedom associated with element nodal points are shown in Table 2 up to 4th order. In general, the element solution can be computed by matching nodes and dofs as shown in Table 2 under the assumption that the solution degrees of blocks are all same. In the present study, the degrees of blocks are assumed different each other, and hence some consideration is necessary on the elements of block interfaces. An example of this kind is shown in Fig. 2. The element with a lower degree is a transition element and the element "A" is such an element. In this case, solution across the elements "A" and "B" should be continuous along the Edge 2 of element "A" and the Edge 4 of element "B". Hence, the edge node on the Edge 4 in "A" should have corresponding degrees of freedom by referring to Table 2. In the case of Fig. 2, the dof number is 6, and if the degree of element "B" is cubic, then the corresponding dofs become 6 and 11.

Usually the number of dof per element node is what is shown in Table 2. If the number of dof of one block is different from that of the other block, the code is arranged for the lower degree element to accept more edge dofs.

Table 2 Dof numbers of element nodes in two dimensions

Deg.	Element Node #								
	1	2	3	4	5	6	7	8	9
1	1	2	3	4					
2					5	6	7	8	9
3					10	11	12	13	14,15, 16
4					17	18	19	20	21,22,23, 24,25

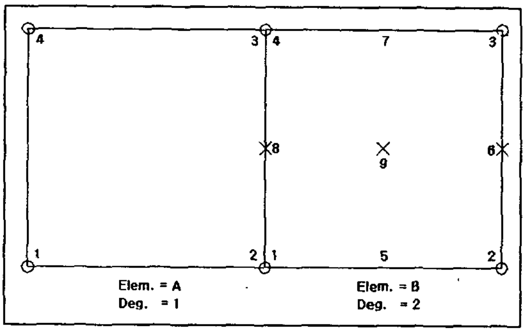

Fig. 2 An Example of transition element

3. Numerical Experiments

To investigate the quality of the solutions from blocks with an arbitrary degree, three simple problems with known exact solutions are chosen[3]. The first one is a one dimensional problems, the second a heat conduction type two dimensional problem and the third a Couette flow problem based on 2-D Stokes equation.

3.1 One dimensional problem

Following equation is considered.

$$-d^2u/dx^2 - u + x^2 = 0, \quad 0 \leq x \leq 1 \qquad (3.1)$$

Two kinds of boundary conditions are considered.

BC(i) : u(0) = 0 = u(1)

BC(ii): u(0) = 0, u'(1) = 1

There are 10 sampling points ranging from x=0.05 to x=0.95. The error norms are plotted in Fig. 3.

Both the linear and the quadratic base show a certain expected convergence trends, but the cubic base does not. Further experiments with 64 elements showed that the quadratic base didn't improve the accuracy of solution with errors remaing at the value of the crossing point appeared in Fig. 3(a). Hence, the error value at the crossing point is the minimum for all method. Moreover, the fact that the cubic base in this case does not improve the solution after 8 elements, might tell that the mesh refinement does not mean more accurate solutions for higher order cases.

From Fig. 3(b), the linear solution shows a certain expected convergence trends, but both the quadratic and the cubic solution do not. More experiments showed that both the quadratic and the cubic solution with 64 elements didn't improve the solution appreciably. A comparison between the exact and the numerical one shows that most of errors of the higher order solutions occurred at the Neumann BC point. In Table 4, errors are shown for each block for the 16 elements solution. This shows that the higher order solutions may not be suitable at the Neumann boundary points, and the errors seem to be reflected inside the domain. The errors in Table 3 and Table 4 are the sum of differences between the numerical and the exact values at the sampling points.

To see the effectiveness of using higher order blocks regionally, 10 numerical tests are made and the results are shown in Table 5. The tests have been done with 4 elements for each block.

The test No. 4 shows that the solution by higher order in the block 4 is worse than those by all other methods shown in Table 4. This phenomena is accrued by using higher order base in the region of Neumann boundary. Considering the test No. 1, the solution is expected, in normal sense, better than those of linear solutions shown in Table 3, but the test shows contrary result. The errors of linear blocks are somewhat lessened but the error of quadratic block is much amplified. In the test No. 5, the degree of block 1 is 3 and the degrees of the other blocks are 1. In this case, the error of block 1 is one order less than that of test No. 1, and errors in the other regions are, up to the digits shown, same as those of the test No. 1.

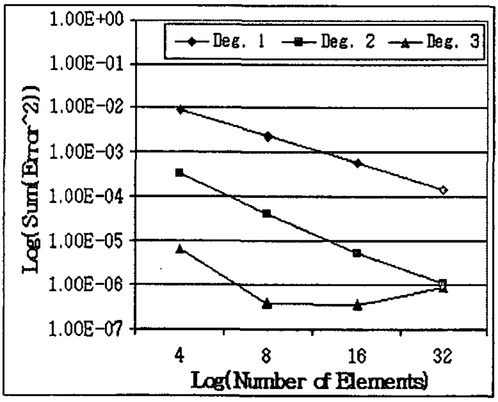

Fig. 3(a) Error plotting for BC type(i)

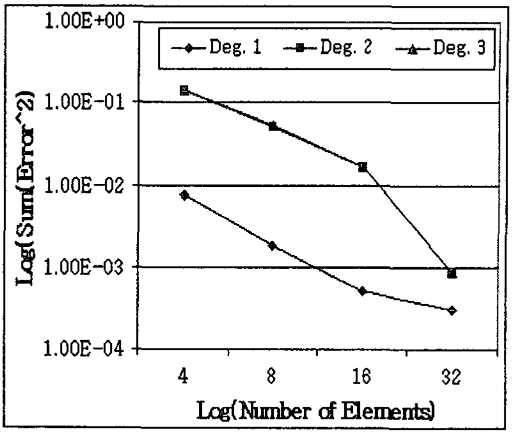

Fig. 3(b) Error plotting for BC type(ii)

The present results may indicate that for one dimensional case, the quadratic block is more sensitive on the error propagation than the other kind of blocks.

The errors shown in the test No. 8 are almost as same as those shown in the test No. 10. This seems that since the exact solution is almost linear except near neumann boundary, the solution by the cubic order block is not much different from the one by the quadratic order block.

In test No. 7, the first two blocks are quadratic and the remaings are linear. In this case the total error is almost same as that of the whole linear case in Table 3.

Table 3 Sum of errors(differences) in each block by BC type (ii) for 16 elements

Block #	Degree		
	1	2	3
1	1.480E-04	5.967E-04	5.952E-04
2	3.985E-04	1.660E-03	1.660E-03
3	4.927E-04	2.702E-03	2.701E-03
4	4.278E-04	1.850E-02	1.850E-02
Sum	1.470E-03	2.350E-02	2.350E-02

In test No. 6, middle of two blocks are quadratic and the blocks near boundaries are linear. The total error shows a good improvement over the whole linear solution.

It has been found that the use of higher order block except the neumann boundary region can increase the overall accuracy of solution in general and also does not affect adversely solution quality of neighboring blocks.

Table 4 Sum of errors(differences) in each block of an arbitrary degree

Test		Block Number			
		1	2	3	4
1	Deg	2	1	1	1
	Err	5.2E-04	3.9E-04	4.9E-04	4.2E-04
2	Deg	1	2	1	1
	Err	1.4E-04	1.1E-04	4.5E-04	3.8E-04
3	Deg	1	1	2	1
	Err	1.3E-04	3.5E-04	1.4E-04	3.5E-04
4	Deg	1	1	1	2
	Err	7.2E-04	2.0E-03	3.1E-03	1.9E-02
5	Deg	3	1	1	1
	Err	5.1E-05	3.9E-04	4.9E-04	4.2E-04
6	Deg	1	2	2	1
	Err	1.2E-04	6.4E-05	9.9E-05	3.0E-04
7	Deg	2	2	1	1
	Err	4.1E-04	1.1E-04	4.5E-04	3.9E-04
8	Deg	2	2	2	1
	Err	2.4E-05	6.3E-05	1.0E-04	3.1E-04
9	Deg	2	2	3	1
	Err	2.4E-05	6.3E-05	1.0E-04	3.1E-04
10	Deg	3	3	3	1
	Err	2.3E-05	6.3E-05	1.0E-04	3.1E-04

3.2 Two dimensional problems

Two problems are chosen: the first is a Poisson type equation and the second a Couette flow problem on a 2-D Stokes equation solved by a primitive variable approach.

(A) Poisson equation

Following equation is considered.

$$-\nabla^2 u = 1, \quad 0 \le x \le 1, \; 0 \le y \le 1, \qquad (3.2)$$

where

$$u = 0 \quad \text{on sides } x=1 \text{ and } y=1$$

$$\partial u / \partial n = 0 \quad \text{on sides } x=0 \text{ and } y=0$$

The exact solution of Eq.(3.2) is found in Ref(3). The number of sampling points ranging from (0.2,0.2) to (0.8,0.8) are 16. Blocks are shown in Fig. 4. Here, three cases are considered. The first case is for one block, the second case for two blocks, and the third for four blocks.

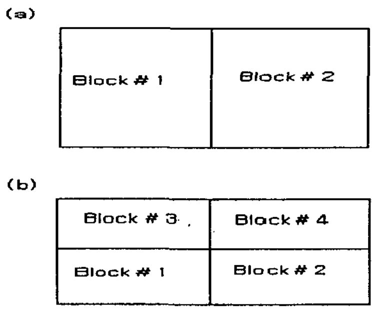

Fig. 4 Block numbering (a) 2 blocks (b) 4 blocks

(1) Solutions by one block

In Fig. 5 they were plotted on a log-log scale. From the plot, one can measure the slopes of each curve with respect to the horizontal line. In the present case, the slope angle for quadratic errors is about one and half of that of the linear one, and the slope angle for cubic is about two times of the linear one.

(2) Solutions by two blocks

Errors(sum of squares) are plotted in Fig. 6. The error plots for Deg(2,1) and Deg(3,1) are not distinguishable. Deg(2,1) means here that the degree of block 1 is quadratic and the degree of block 2 is linear. The slope angle for Deg(3,2) is about two times for that of Deg(2,1) or Deg(3,1) up to 64 elements, but it becomes almost same as those from the others for more elements than 64. With higher order elements in block 1, the solution is much improved from the pure linear element case. With higher order elements for all blocks, the initial slope is almost twice those of the other case, but as elements are refined, the convergence rate gets slower. This tells that for higher order elements, there might be a limitation on mesh refinement.

Overall, the use of higher order block improves the solution as was expected.

(3) Solutions by four blocks

Fig. 7 shows errors for four different kinds of mixed degrees. The error for Deg(2,1,1,1) and Deg(3,1,1,1) are almost same and the solution by Deg(3,2,2,2) converges more rapidly. The solution by Deg(3,2,2,1) is expected to show a similar behavior to that by Deg(3,2,2,2), but is not. This may be that if there is a linear block, the overall convergence is very much dependent on the fashion inherent to linear elements.

It may be of interest to see how the higher order block affects neighboring blocks, and for this, Table 5 was prepared by running with 64 elements, ie, 4 by 4 elements for each "domain". Referring to two block method in Table 5, errors in domain 2 and 4 of both Deg(2,1) and Deg(3,1) are greater than that of Deg(1) of one block method. It may be said that, when using mixed degrees, there exists an intraboundary where numerical errors are reflected. Also when using the present error norm, it turns out that the Deg(2,1) solution is worse than Deg(1) solution, which is quite contradictory.

From four blocks case, the behavior of Deg(2,1,1,1) solution looks quite ordinary since this case reduces errors in domain 1 and does not affect the other domains adversely. Deg(3,1,1,1) solution has similar behavior but is not improved as expected. Deg(3,2,2,1) solution looks better but actually the sum of errors are greater than that of Deg(1) solution. This shows that errors are accumulated inside the linear block severely. The solution of Deg(3,2,2,2)

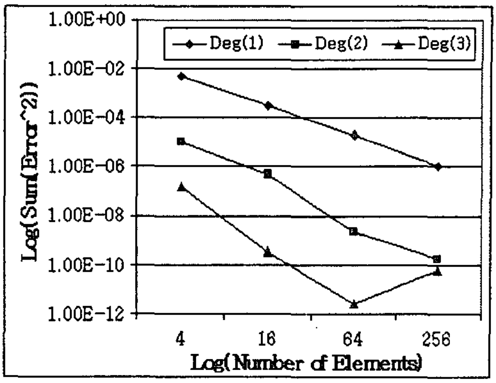

Fig. 5 Error plots for two dimensional poisson problem

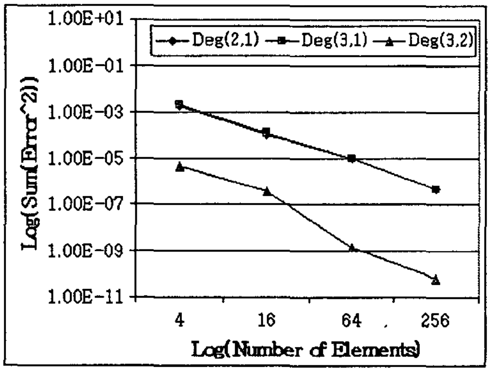

Fig. 6 Error plots for two block method

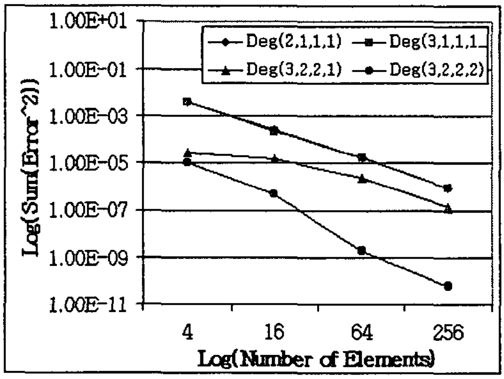

Fig. 7 Error plots for four block method

shows a better solution than that of Deg(2). This solution does not exhibit any error accumulation comparing with the Deg(2) solution. Hence it can be concluded that when using mixed degree elements with linear elements, the region of higher order elements should be small in order to have enough dofs in linear base block to damp out errors.

Table 5 Sum of errors(differences) in each domain for 64 elements

Domain.	One Block		
	Deg(1)	Deg(2)	Deg(3)
1	3.098E-03	2.850E-05	4.000E-07
2	4.421E-03	3.350E-05	4.000E-07
3	4.421E-03	3.350E-05	4.000E-07
4	4.672E-03	5.072E-05	2.070E-06
Sum	1.661E-02	1.462E-04	3.270E-06

Domain.	Two Blocks		
	Deg(2,1)	Deg(3,1)	Deg(3,2)
1	1.808E-03	2.084E-03	9.500E-06
2	1.687E-02	1.650E-02	2.817E-04
3	1.409E-03	1.117E-03	1.280E-05
4	1.339E-02	1.442E-02	9.111E-04
Sum	3.348E-02	3.412E-02	1.215E-03

Domain	Four Blocks			
	Deg (2,1,1,1)	Deg (3,1,1,1)	Deg (3,2,2,1)	Deg (3,2,2,2)
1	8.66E-04	8.50E-04	2.22E-05	1.60E-06
2	4.28E-03	4.30E-03	3.23E-05	3.49E-05
3	4.28E-03	4.30E-03	3.22E-05	3.50E-05
4	4.76E-03	4.76E-03	3.04E-02	5.09E-05
Sum	1.42E-02	1.42E-02	3.05E-02	1.22E-04

(B) Couette Flow problem on 2-D Stokes equation

The 2-D Stokes equation is as follows:

$$\begin{aligned} -\nu \nabla^2 u + \partial p/\partial x &= 0 \\ -\nu \nabla^2 v + \partial p/\partial y &= 0 \\ \partial u/\partial x + \partial v/\partial y &= 0 \end{aligned} \qquad (3.3)$$

The domain of computation is $0 \le x \le 2$, $0 \le y \le 6$, and the boundary conditions are shown in Fig. 8. The main difficulity of this problem is that the pressure solution exhibits an hourglass fashion. To avoid the hourglass phenomena, it is a standard practice to use one order less interpolation for the pressure. In the present study, pressures are computed only at the vertex dofs, which means that the values on edges and interior for the pressures are linearly interpolated. Thus the edge and the interior dofs for the pressure function become zeros since the hierarchical functions other than vertices are related with dofs having second or more tangential derivatives. Table 6 shows the sum of squared errors on u,v and p for both quadratic and cubic base with mesh refined.

With pressure interpolated by the same order as for the velocities, the errors in pressure(elem. number of 2 by 6 and 4 by 12) are order of -7 without showing a chekerboard manner. This is accrued to the loose connection between the vertex dof and the other dofs in the hierarchic base functions. However, as the elements are refined to 8 by 24, the error order

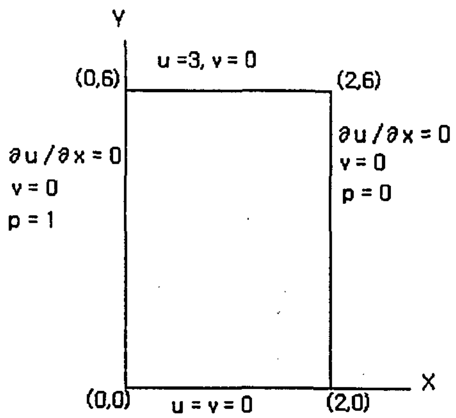

Fig. 8 Domain and boundary conditions for Couette flow problem

Table 6 Sum of errors for the Coutte flow problem

Elem. #	Deg.	Error in U	Error in V	Error in P
2 by 6	2	1.325E-11	9.473E-14	1.069E-12
	3	4.975E-11	6.991E-14	1.023E-12
4 by 12	2	1.127E-11	3.557E-13	9.946E-09
	3	2.969E-10	9.207E-13	1.541E-11
8 by 24	2	3.825E-10	7.485E-13	1.078E-10
	3	2.794E-09	7.579E-12	4.960E-11
4 by 12	(3,2,2)	6.166E-10	2.709E-13	4.797E-12
	(2,3,2)	1.879E-10	6.426E-13	4.928E-12
	(2,2,3)	4.095E-11	1.845E-13	8.184E-12

becomes -5 and begins to show the chekerboard type. In this case, the mesh refinement does not improve the solution notably.

For mixed degree solutions, the errors are also shown in Table 6(for 4 by 12 elements). In this case, domain is devided into 3 blocks along the vertical direction. The bottom block is numbered as one, the middle as two and the uppermost as three. Even though the use of cubic degree for the velocities does not show appreciable improvements in solution qualities, the Deg(2,2,3) solution shows some improvement in velocities at some expense on the pressure solution.

5. Conclusions

To investigate the efficiency of higher order elements, some numerical tests with hierarchical shape functions and mixed order in a domain are conducted, and the observations are presented.

This study draws following four conclusions.

1) By using hierarchical shape functions, one can generate mixed order solutions in a systematic way and can code in an efficient manner.
2) For higher order regions, mesh refining does not always give more accurate solutions, hence a proper mesh size is essential.
3) The higher order solutions are very much dependent on the type of boundary conditions, especially for the hierarchical elements. Hence it would be safer to use away from the boundary regions.
4) When using higher order elements as a remedy on a linear element solution, the region of linear elements shoule be large enough to damp out numerical errors.

The present study, before going on a construction of three dimensional elements, summarizes some test results on two dimensional cases to reevaluate the present code. Since the code is developed based on the assembly of each simple blocks, it seems to be well adapted to a modern solver such as a domain decomposition method. Also it is desired to develop a solver to take advantage of orthogonality properties of hierarchical shape functions.

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