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SOME GEOMETRIC APPLICATIONS OF EXTREMAL LENGTH (I)

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ABSTRACT. In this note, we present some geometric applications of extremal length to analytic functions. We drive an interesting formula by the method of extremal length.

1. Extremal length

The method of extremal length has great advantages in the theory of analytic functions, among others it applies with almost equal facility to problems on simply - and multiply connected domains.

Throughout this note, we are working over the finite complex plane $\mathbb{C}$. Let Γ be a family of curves in a domain D, and ρ be a non-negative Borel measurable function defined on $\mathbb{C}$.

For $L(\Gamma, \rho) = \inf\limits_{\gamma \in \Gamma} \int_\gamma \rho |dz|$ and $A(D, \rho) = \iint_D \rho^2 dxdy$, the *extremal length* of Γ in D is defined as

$$\lambda_D(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(D, \rho)}. \tag{1}$$

$\lambda_D(\Gamma)$ depends only on Γ and not on D. Accordingly, we shall simplify the notation to $\lambda(\Gamma)$ instead of $\lambda_D(\Gamma)$ ([1]).

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PROPOSITION 1.1. ([2]) *Let* Δ *be the annulus* $\Delta = \{z|a < |z| < b\}$. *Let* Γ *be the family of all curves in* Δ *which join the two contours. Then*

$$\lambda(\Gamma) = \frac{1}{2\pi}\log\frac{b}{a}.$$

proof. In fact, for any ρ, we have

$$\int_a^b \rho\, dr \geq L(\Gamma,\rho), \quad \iint_\Delta \rho\, dr d\theta \geq 2\pi L(\Gamma,\rho).$$

Then, by the Schwarz inequality, (ref. [3]),

$$\begin{aligned} 4\pi^2 L^2(\Gamma,\rho) &\leq (\iint_\Delta \rho\, dr\, d\theta)^2 \\ &\leq \iint_\Delta \rho^2 \frac{1}{r}\, dr\, d\theta \iint_\Delta r\, dr\, d\theta \\ &= 2\pi \log\frac{b}{a} \iint_\Delta \rho^2\, r\, dr\, d\theta. \end{aligned}$$

This proves $\lambda(\Gamma) \leq \frac{1}{2\pi}\log\frac{b}{a}$.

On the other hand, take $\rho = \frac{1}{r}$.

$$L(\Gamma, \frac{1}{r}) = \log\frac{b}{a}, \qquad A(\Delta, \frac{1}{r}) = 2\pi\log\frac{b}{a},$$

hence $\lambda(\Gamma) \geq \frac{1}{2\pi}\log\frac{b}{a}$. □

PROPOSITION 1.2. ([1]) (Comparison principle of extremal length) *For two curve families* Γ_1, Γ_2, *if every* $\gamma_2 \in \Gamma_2$ *contains a* $\gamma_1 \in \Gamma_1$, *then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2).$$

proof. Indeed, both extremal lengths can be evaluated with respect to the same D. For any ρ in D it is clear that $L(\Gamma_2,\rho) \geq L(\Gamma_1,\rho)$. These minimum lengths are compared with the same $A(D,\rho)$. □

Briefly, the set Γ_2 of fewer or longer curves has the larger extremal length.

The conformal invariance of extremal length is an immediate consequence of the definition.

PROPOSITION 1.3. ([5]) (Conformal invariance of extremal length) Let $z^* = f(z)$ be a 1-1 conformal mapping on D upon a domain D^* and Γ be a family of curves in D, then

$$\lambda(\Gamma) = \lambda[f(\Gamma)].$$

2. Geometric application

Here we shall give an alternative simple proof of the interesting formula. The function- theoretic proof of this formula is difficult. The use of extremal length makes the proof trivial.

It is a consequence of the Riemann mapping theorem that any two simply connected proper subdomains of the plane are conformally equivalent. One may ask whether any two annuli are conformally equivalent. The answer is negative.

THEOREM A. *Let* $\Delta(r, R) = \{z | \ r < |z| < R\}, (0 < r < R < \infty)$. *Then* $\Delta_1(r_1, R_1)$ *and* $\Delta_2(r_2, R_2)$ *are conformally equivalent if and only if*

$$\frac{R_1}{r_1} = \frac{R_2}{r_2}. \tag{2}$$

Alternative short Proof. If $\dfrac{R_1}{r_1} = \dfrac{R_2}{r_2}$, the mapping $f(z) = \frac{r_2}{r_1} z$ maps Δ_1 onto Δ_2. Hence Δ_1 and Δ_2 are conformally equivalent.

Let Γ_Δ be the family of all curves in Δ which join the two contours. Then by Proposition 1.1,

$$\lambda(\Gamma_\Delta) = \frac{1}{2\pi} \log \frac{R}{r}. \tag{3}$$

Suppose that Δ_1 and Δ_2 are conformally equivalent and let f be a 1-1 conformal mapping on Δ_1 upon Δ_2. Then by the conformal invariance of extremal length,

$$\lambda(\Gamma_{\Delta_1}) = \lambda[f(\Gamma_{\Delta_1})] = \lambda(\Gamma_{\Delta_2}). \tag{4}$$

Hence by (3) and (4), we obtain(2). □

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