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ON THE RANGE OF DERIVATIONS

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ABSTRACT. In this paper we will show that if $[G(y), x]D(x)$ lies in the nil radical of A for all $x \in A$, then GD maps A into the radical, where D and G are derivations on a Banach algebra A.

1. Introduction

Let A be an algebra over a complex field $\mathbb{C}$. The Jacobson radical of A and the nil radical of A will be denoted by $rad(A)$ and $nil(A)$, respectively. We write $[x,y]$ for $xy-yx$, and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that an algebra A is prime if $aAb = \{0\}$ implies that either $a = 0$ or $b = 0$. A linear mapping D from A to A is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$.

Singer and Wermer [6] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. They also made a very insightful conjecture, namely that the assumption of continuity was unnecessary. This became known as the Singer-Wermer conjecture and was proved in 1988 by Thomas [7]. The so-called noncommutative Singer-Wermer conjecture was proved that every derivation D on a Banach algebra A such that $[D(x), x] \in rad(A)$ for all $x \in A$ maps the algebra into its radical. As an evidence for the validity of the conjecture, Mathieu showed that the above conclusion

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holds if the condition $[D(x), x] \in rad(A)$ for all $x \in A$ is replaced by the condition $[D(x), x] \in nil(A)$ for all $x \in A$ [3, Theorem 1]. In this paper we will show that the condition $[G(y), x]D(x) \in nil(A)$ for all $x, y \in A$ also guarantees the result of Mathieu.

2. The Results

To prove our main theorem, we shall need the following purely algebraic result.

LEMMA 2.1. *Let D and G be derivations on a noncommutative prime algebra A such that $[G(y), x]D(x) = 0$ for all $x, y \in A$. Then we have either $D = 0$ or $G = 0$.*

Proof. Suppose that

$$[G(y), x]D(x) = 0 \tag{1}$$

for all $x, y \in A$. Taking $y = yD(x)$ in (1), we obtain

$$G(y)[D(x), x]D(x) + [y, x]G(D(x))D(x) = 0 \tag{2}$$

for all $x, y \in A$. Replacing y by xy in (2), we have

$$G(x)y[D(x), x]D(x) = 0 \tag{3}$$

for all $x, y \in A$. Since A is prime, we get either $G(x) = 0$ or $[D(x), x]D(x) = 0$ for any $x \in A$. Thus A is the union of its subsets $A_1 = \{x \in A : G(x) = 0\}$ and $A_2 = \{x \in A : [D(x), x]D(x) = 0\}$.

Suppose $G \neq 0$ and $D \neq 0$. The principal results in [9] then tell us that $A_1 \neq A$ and $A_2 \neq A$. Thus there exist $x, y \in A$ such that $x \notin A_1$ and $y \notin A_2$. Hence $x \in A_2$ and $y \in A_1$. If we consider $x + \lambda y$ for all $\lambda \in \mathbb{C}$, then we see that either $x + \lambda y \in A_1$ or $x + \lambda y \in A_2$. In case $x + \lambda y \in A_1$, we have

$$G(x) = 0. \tag{4}$$

In case $x + \lambda y \in A_2$, we get

(5)
$$\begin{aligned}&\lambda\{[D(x),x]D(y)+[D(x),y]D(x)+[D(y),x]D(x)\}\\&\quad+\lambda^2\{[D(x),y]D(y)+[D(y),x]D(y)+[D(y),y]D(x)\}\\&\quad+\lambda^3[D(y),y]D(y)=0.\end{aligned}$$

Thus one of these two possibilities holds. But (5) has more than three solutions. This contradicts the assumptions that $G(x) \neq 0$ and $[D(y),y]D(y) \neq 0$. This completes the proof. □

THEOREM 2.2. *Let D and G be continuous derivations on a Banach algebra A such that $[G(y),x]D(x) \in rad(A)$ for all $x, y \in A$. Then we have $GD(A) \subseteq rad(A)$.*

Proof. Let J be a primitive ideal of A. Since D and G are continuous, by [4, Theorem 2.2], we have $D(J) \subseteq J$ and $G(J) \subseteq J$. Then we can define derivations D_J and G_J on A/J by $D_J(x+J) = D(x) + J$, $G_J(x+J) = G(x) + J$ for all $x \in A$. The factor algebra A/J is prime and semisimple, since J is a primitive ideal. Johnson and Sincliar [2] have proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative Banach algebra. Hence in case A/J is commutative, we have both $D_J = 0$ and $G_J = 0$. It remains to show that either $D_J = 0$ or $G_J = 0$ in the case when A/J is noncommutative. Note that the intersection of all primitive ideals is the radical. The assumption of the theorem

$$[G(y),x]D(x) \in rad(A)\ (x, y \in A)$$

gives

$$[G_J(y+J), x+J]D_J(x+J) = J\ (x, y \in A).$$

All the assumption of lemma 2.1 is fulfilled. Thus we have $G_J D_J = 0$ in any case . Hence we see that $GD(A) \subseteq J$ since J is a primitive ideal. This completes the proof. □

From the above result, we prove the following theorem.

THEOREM 2.3. *Let D and G be derivations on a Banach algebra A such that $[G(y), x]D(x) \in nil(A)$ for all $x, y \in A$. Then we have $GD(A) \subseteq rad(A)$.*

Proof. Let J be any primitive ideal of A. Using Zorn's lemma, we can find a minimal prime ideal P contained in J, and hence $D(P) \subseteq P$ and $G(P) \subseteq P$. Suppose first that P is closed. Then derivations D and G on a Banach algebra A induce derivations $\bar{D}$ and $\bar{G}$ on a Banach algebra A/P defined by

$$\bar{D}(x+P) = D(x) + P,\ \bar{G}(x+P) = G(x) + P$$

for all $x \in A$. In case A/P is commutative, $\bar{D}(A/P)$ and $\bar{G}(A/P)$ are contained in the radical of A/P by [7]. In case A/P is noncommutative, Lemma 2.1 implies that either $\bar{D} = 0$ or $\bar{G} = 0$ on A/P, since A/P is prime and $[\bar{G}(y+P), x+P]\bar{D}(x+P) = P$ for all $x, y \in A$. In both cases, $\bar{G}\bar{D}(A/P) \subseteq J/P$. Consequently we see that $GD(A) \subseteq J$. If P is not closed, then we see that $S(D) \subseteq P$ by [1, Lemma 2.3] (where $S(D)$ is the separating space of D). Then, by [5, Lemma 1.3], we have $S(Q_{\bar{P}}D) = \overline{Q_{\bar{P}}(S(D))} = \{0\}$, where $Q_{\bar{P}}$ is the quotient map from A to $A/\bar{P}$. From this $Q_{\bar{P}}D$ is continuous. Therefore $Q_{\bar{P}}D(\bar{P}) = \{0\}$ on $A/\bar{P}$, that is, $D(\bar{P}) \subseteq \bar{P}$. In the same fashion, $G(\bar{P}) \subseteq \bar{P}$. From this we can also define continuous derivations $\widetilde{D}$ and $\widetilde{G}$ on $A/\bar{P}$ by

$$\widetilde{D}(x+\bar{P}) = D(x) + \bar{P},\ \widetilde{G}(x+\bar{P}) = G(x) + \bar{P}$$

for all $x \in A$. Hence $[\widetilde{G}(y+\bar{P}), x+\bar{P}]\widetilde{D}(x+\bar{P})$ is contained in the radical of $A/\bar{P}$. By Theorem 2.2 $\widetilde{G}\widetilde{D}(A/\bar{P})$ is contained in the radical of $A/\bar{P}$. Thus $\widetilde{D}(A/\bar{P}) \subseteq J/\bar{P}$ or $\widetilde{G}(A/\bar{P}) \subseteq J/\bar{P}$. Therefore $GD(A) \subseteq J$. This completes the proof. □

The below result is an immediate consequence of Theorem 2.3.

COROLLARY 2.4. *Let D and G be derivations on a semisimple Banach algebra A such that $[G(y), x]D(x) = 0$ for all $x, y \in A$. Then we have $GD = 0$.*

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