JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 12, August 1999

UNIFORMLY FUZZY CONTINUOUS ON THE FUZZY REAL LINE

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ABSTRACT. In this paper, we shall define the usual fuzzy distance between two real fuzzy points, using the usual distance between two points in $\mathbb{R}$. We introduce the fuzzy sequence in the fuzzy real line and the notion of limit of fuzzy sequence in $F_p(\mathbb{R})$, and obtain uniformly fuzzy continuous in $F_p(\mathbb{R})$.

1. Introduction

Throughout this paper, we denote the closed interval $[0,1]$ by I, while $I_0 = (0,1]$ and $\mathbb{R}^+ = [0,\infty)$. A fuzzy set A in the set X is characterized by a membership function μ_A from X to I. The set I^X is the set of all fuzzy sets in X. Two fuzzy sets A and B are said to be equal iff $\mu_A(x) = \mu_B(x)$ for all $x \in X$. The support of $A \in I^X$, denoted by $S(A)$, is the ordinary subset of X, that is $S(A) = \{x \in X | \mu_A(x) > 0\}$. The union and intersection of $\{A_j \in I^X | j \in J\}$, denoted by $\cup_{j\in J} A_j$ and $\cap_{j \in J} A_j$ respectively, are defined by the membership functions

$$\mu_{\cup_{j\in J} A_j}(x) = \bigvee_{j\in J} \mu_{A_j}(x),$$
$$\mu_{\cap_{j\in J} A_j}(x) = \bigwedge_{j\in J} \mu_{A_j}(x) \quad \text{for all} \quad x \in X.$$

A is said to be included in B, denoted by $A \subseteq B$, iff $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

Received by the editors on June 28, 1999.
1991 *Mathematics Subject Classifications*: Primary 26E50.
Key words and phrases: Fuzzy sequence, Fuzzy continuous.

A fuzzy point in $\mathbb{R}$ is a fuzzy set in $\mathbb{R}$ which is zero everywhere except at the one point, say x, where it takes a value , say α , in I_0. This fuzzy point is denoted by x_α, which is called a fuzzy point with support x and the value α. The collection of all fuzzy points in $\mathbb{R}$ will be denoted by $F_p(\mathbb{R})$. A fuzzy point x_α is said that x_α is an element of A, denoted by $x_\alpha \in A$ iff $\alpha < \mu_A(x)$.

If $x_\alpha \in F_p(\mathbb{R})$, the *fuzzy absolute value* of x_α, denoted by $|x_\alpha|$, is defined by

$$|x_\alpha| = \begin{cases} x_\alpha & \text{if} \quad x \geq 0, \\ (-x)_\alpha & \text{if} \quad x < 0. \end{cases}$$

Define a relation $\leq$ in $F_p(\mathbb{R})$ by for any $x_\alpha, y_\beta \in F_p(\mathbb{R})$, $x_\alpha \leq y_\beta$ if $x < y$ or $x = y$ and $\alpha \leq \beta$. The relation $\leq$ in $F_p(\mathbb{R})$ satisfies that for any $x_\alpha, y_\beta, z_\gamma \in F_p(\mathbb{R})$,

(1) $x_\alpha \leq x_\alpha$,

(2) $x_\alpha \leq y_\beta$ and $y_\beta \leq x_\alpha$ imply $x_\alpha = y_\beta$,

(3) $x_\alpha \leq y_\beta$ and $y_\beta \leq z_\gamma$ imply $x_\alpha \leq z_\gamma$.

We call that the relation $\leq$ in $F_p(\mathbb{R})$ is the *usual fuzzy order*. In particular, we write $x_\alpha \ll y_\beta$ if $x < y$ and $\alpha < \beta$.

For a metric space (X, D), a fuzzy distance $\tilde{D}$ between fuzzy sets A and B in X is defined using D as

$$\mu_{\tilde{D}(A,B)}(\delta) = \bigvee_{\delta = D(u,v)} (\mu_A(u) \wedge \mu_B(v)) \quad \text{for all} \quad \delta \in \mathbb{R}^+,$$

where $\tilde{D}$ is a mapping from $[\tilde{P}(X)]^2$ to $\tilde{P}(\mathbb{R}^+)$.

We define a distance function $d : F_p(X) \times F_p(X) \to \tilde{P}(\mathbb{R}^+)$ by the restriction of $\tilde{D}$ to $[F_p(X)]^2$. Note that each pair (x_α, y_β) in $[F_p(X)]^2$ corresponds to the fuzzy point $D(x,y)_{\alpha\wedge\beta}$ with support $D(x,y)$ and the value $\alpha \wedge \beta$.

DEFINITION 1.1. [3] The *usual fuzzy metric* $d : F_p(\mathbb{R}) \times F_p(\mathbb{R}) \to \tilde{P}(\mathbb{R}^+)$ is defined by $d(x_\alpha, y_\beta) = |x - y|_{\alpha\wedge\beta}$ for every $(x_\alpha, y_\beta) \in F_p(\mathbb{R}) \times F_p(\mathbb{R})$. We call the pair $(\mathbb{R}, d)$ the usual fuzzy metric space.

DEFINITION 1.2. [3] The open fuzzy ball $B(x_\alpha; r_\alpha)$ with center x_α and radius r_α is the fuzzy set

$$B(x_\alpha; r_\alpha) = \bigcup \{y_\beta \in F_p(\mathbb{R}) : d(x_\alpha, y_\beta) \ll r_\alpha\},$$

where $|x - y|_{\alpha \wedge \beta} \ll r_\alpha$ means that $|x - y| < r$ and $\alpha \wedge \beta < \alpha$.

We see that $S[B(x_\alpha; r_\alpha)] = (x - r, x + r)$ and $\mu_{B(x_\alpha; r_\alpha)}(y) = \alpha$ for all $y \in (x - r, x + r)$. As in the notation of a fuzzy point we denote $B(x_\alpha; r_\alpha)$ by $(x - r, x + r)_\alpha$ and call it the *open fuzzy interval with the value* α .

A fuzzy set A in $\mathbb{R}$ in called an *fuzzy open interval with the value* α for $\alpha \in I_0$ or simply *fuzzy open interval* iff $S(A)$ is an open interval (a, b) and

$$\mu_A(x) = \begin{cases} \alpha & \text{if} \quad x \in S(A), \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we shall denote by $(a, b)_\alpha$. Similarly, we can define the other fuzzy intervals with the value α, $[a, b]_\alpha, [a, b)_\alpha, (a, b]_\alpha$, where $a = -\infty$ and $b = \infty$ are admissible.

DEFINITION 1.3. [3] A fuzzy set A in $\mathbb{R}$ is called an *open fuzzy set* if and only if for every $x \in S(A)$ and for every $0 \leq \lambda < \mu_A(x)$ there exits an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon)_\lambda \subset A$.

PROPOSITION 1.1. *Let $\mathcal{U}$ be the family of all open fuzzy set in $\mathbb{R}$. Then $\mathcal{U}$ satisfies the following:*

(1) *For each $\alpha \in I$, $(-\infty, \infty)_\alpha \in \mathcal{U}$, where $(-\infty, \infty)_0$ means the empty set.*
(2) *If $\{A_i \in \mathcal{U} | i \in I\}$, then $\cup_{i \in I} A_i \in \mathcal{U}$.*
(3) *If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.*

The family $\mathcal{U}$ in the above proposition is called the *usual fuzzy topology* for $\mathbb{R}$ and the pair $(\mathbb{R}, \mathcal{U})$ the *usual fuzzy topological space.*

DEFINITION 1.4. Let $(\mathbb{R},\mathcal{U})$ be a usual fuzzy topological space, and $x_\lambda \in F_p(\mathbb{R})$. Then a mapping $f : F_p(\mathbb{R}) \to F_p(\mathbb{R})$ is said to be *fuzzy continuous* at x_λ if for each open neighborhood V of $f(x_\lambda)$, there exists an open neighborhood U of x_λ such that $f(U) \subset V$. And the mapping f is said to be fuzzy continuous on $\mathbb{R}$ if it is fuzzy continuous at each $x_\lambda \in F_p(\mathbb{R})$.

DEFINITION 1.5. [4] A usual fuzzy real sequence $< x^{(n)}_{\alpha_n} >$ *converges* to a fuzzy point x_α if $x^{(n)} \to x$ and $\alpha_n \to \alpha$, that is $d(x^{(n)}_{\alpha_n}, x_\alpha) \to 0_\alpha$. Thus, we have that $x^{(n)}_{\alpha_n} \to x_\alpha$ if for any given $\epsilon > 0$ there exists a natural number N such that $n > N$ implies $|x^{(n)} - x| < \epsilon$ and $|\alpha_n - \alpha| < \epsilon$. In this case, x_α is called the *limit* of $< x^{(n)}_{\alpha_n} >$.

PROPOSITION 1.2. *A fuzzy sequence can converge to at most one fuzzy point of* $\mathbb{R}$.

PROPOSITION 1.3. *Let* $< x^{(n)}_{\alpha_n} > \to x_\alpha$ *and* $< y^{(n)}_{\beta_n} > \to y_\beta$. *Then, we have the followings;*

(1) $< x^{(n)}_{\alpha_n} \pm y^{(n)}_{\beta_n} >$ *converges to* $x_\alpha \pm y_\beta$.
(2) *For any given* $k_\gamma \in F_p(\mathbb{R})$, $< k_\gamma x^{(n)}_{\alpha_n} >$ *converges to* $(kx)_{\gamma\wedge\alpha}$.
(3) $< x^{(n)}_{\alpha_n} y^{(n)}_{\beta_n} >$ *converges to* $(xy)_{\alpha\wedge\beta}$.
(4) *Let* $y^{(n)} \neq 0$ *for all* $n \in \mathbb{N}$ *and let* $y \neq 0$. *Then,* $< x^{(n)}_{\alpha_n} / y^{(n)}_{\beta_n} >$ *converges to* $(x/y)_{\alpha\wedge\beta}$.

Here, $x^{(n)}_{\alpha_n} \pm y^{(n)}_{\beta_n} = (x^{(n)} \pm y^{(n)})_{\alpha_n\wedge\beta_n}$, $x^{(n)}_{\alpha_n} y^{(n)}_{\beta_n} = (x^{(n)} y^{(n)})_{\alpha_n\wedge\beta_n}$ *and* $x^{(n)}_{\alpha_n} / y^{(n)}_{\beta_n} = (x^{(n)}/y^{(n)})_{\alpha_n\wedge\beta_n}$, $y^{(n)} \neq 0$.

COROLLARY 1.4. *A fuzzy set* A *in* $\mathbb{R}$ *is bounded iff* $S(A)$ *is bounded in ordinary set* $\mathbb{R}$.

DEFINITION 1.6. A usual real fuzzy sequence $< x^{(n)}_{\alpha_n} >$ is said to be *bounded* if $< x^{(n)} >$ is bounded, i.e., there exists a real number $M > 0$ such that $|x^{(n)}| \leq M$ for all $n \in \mathbb{N}$.

PROPOSITION 1.5. *A convergent sequence of real fuzzy points is bounded.*

REMARK. A sequence $< x_{\alpha_n}^{(n)} >$ is bounded if and only if the set $\{x_{\alpha_n}^{(n)} | n \in N\}$ is bounded in $F_p(\mathbb{R})$.

2. Uniformly fuzzy continuous

DEFINITION 2.1. [5] A function f from a subset D of $F_p(\mathbb{R})$ into $F_p(\mathbb{R})$ is called a *usual fuzzy real function.*

Since we may identify a fuzzy point x_α in $\mathbb{R}$ with the point (x, α) in the plane, it is natural to define the following.

DEFINITION 2.2. [5] A subset C of $F_p(\mathbb{R})$ is called a *usual fuzzy curve* if the subset $\{(x, \alpha) | x_\alpha \in C\}$ of $\mathbb{R}^2$ is a curve, i.e., there is a continuous function c from an interval $[a, b]$ in $\mathbb{R}$ onto $\{(x, \alpha) | x_\alpha \in C\}$ in $\mathbb{R}^2$. In this case, we say that C is a *curve* from the fuzzy point $c(a)$ to the fuzzy point $c(b)$ or $c(a)$ and $c(b)$ are joined by C.

DEFINITION 2.3. [5] A subset D of $F_p(\mathbb{R})$ is said to be *fuzzily connected* if every pair of fuzzy points x_α and y_β in D can be joined by a curve that lies entirely in D.

Note that we have

$$D_f = \bigcup_{x \in S(D_f)} \Big(\bigcup_{0 < \lambda \leq \mu_{D_f}(x)} x_\lambda \Big),$$

where $S(D_f)$ is the support of D_f. In this reason, we may identify a fuzzy set D_f in $\mathbb{R}$ with the subset $D = \{x_\lambda | x \in S(D_f), 0 < \lambda \leq \mu_{D_f}(x)\}$ of $F_p(\mathbb{R})$.

DEFINITION 2.4. A subset D of $F_p(\mathbb{R})$ is called a (fuzzy) *domain* if it is connected and there is a fuzzy open set D_f such that $D = \{x_\lambda | x \in S(D_f), 0 < \lambda \leq \mu_{D_f}(x)\}$. D is called a *fuzzy open(closed) domain* if $S(D_f)$ is a open(closed) interval in $\mathbb{R}$.

In $\mathbb{R}^2$, we say that (x, y) approaches (a, b), written $(x, y) \to (a, b)$ if x approaches a and y approaches b. In this reason, it is natural to say that x_λ approaches a_α, written $x_\lambda \to a_\alpha$ if $x \to a$ and $\lambda \to \alpha$.

DEFINITION 2.5. Let D be a (fuzzy) domain and let f be a function from D into $F_p(\mathbb{R})$. We say that f has the limit $l_\beta \in F_p(\mathbb{R})$ as x_λ approaches a_α, written $\lim_{x_\lambda \to a_\alpha} f(x_\lambda) = l_\beta$ or $f(x_\lambda) \to l_\beta$ as $x_\lambda \to a_\alpha$, provided that for any given $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta, |\lambda - \beta| < \delta$ and $x_\lambda \in D - a_\alpha$ imply $|S(f(x_\lambda)) - l| < \epsilon$ and $|V(f(x_\lambda)) - \beta| < \epsilon$, where $V(f(x_\lambda))$ is the value of the fuzzy point $f(x_\lambda)$.

PROPOSITION 2.1. *Let D be a domain and let $f, g : D \to F_p(\mathbb{R})$. If $\lim\limits_{x_\lambda \to a_\alpha} f(x_\lambda) = s_\mu$ and $\lim\limits_{x_\lambda \to a_\alpha} g(x_\lambda) = t_\nu$,*

(1) $\lim\limits_{x_\lambda \to a_\alpha} [f(x_\lambda) + g(x_\lambda)] = s_\mu + t_\nu = (s + t)_{\mu \wedge \nu}$.
(2) $\lim\limits_{x_\lambda \to a_\alpha} [f(x_\lambda) - g(x_\lambda)] = s_\mu - t_\nu = (s - t)_{\mu \wedge \nu}$.
(3) *For any given l_θ,* $\lim\limits_{x_\lambda \to a_\alpha} [l_\theta f(x_\lambda)] = l_\theta s_\mu = (ls)_{\theta \wedge \mu}$.
(4) $\lim\limits_{x_\lambda \to a_\alpha} [f(x_\lambda) g(x_\lambda)] = s_\mu t_\nu = (st)_{\mu \wedge \nu}$.
(5) *Let $0 \notin S(g(D))$ and let $t \neq 0$. Then,* $\lim\limits_{x_\lambda \to a_\alpha} [f(x_\lambda)/g(x_\lambda)] = s_\mu / t_\nu = (s/t)_{\mu \wedge \nu}$.

REMARK. Let D be a domain and $f : D \to F_p(\mathbb{R})$. We define that f is *fuzzy continuous* at $a_\alpha \in D$ if $\lim_{x_\lambda \to a_\alpha} f(x_\lambda) = f(a_\alpha)$.

DEFINITION 2.6. A function $f : D \to F_p(\mathbb{R})$ is said to be *bounded* on D if $S(f(D))$ is bounded on $\mathbb{R}$. That is, there is $M > 0$ such that $|S(f(x_\lambda))| \leq M$ for all $x_\lambda \in D$.

A mapping is bounded if its range is a bounded fuzzy set in $F_p(\mathbb{R})$.

THEOREM 2.2. *Let D be a fuzzy closed bounded domain and let $f : D \to F_p(\mathbb{R})$ be fuzzy continuous on D. Then f is bounded on D.*

Proof. Suppose that f is not bounded on D. Then, for any $n \in \mathbb{N}$, there is a fuzzy point $x^{(n)}_{\alpha_n} \in D$ such that $|S(f(x^{(n)}))| > n$. Since D is bounded, the fuzzy sequence $< x_{\alpha_n}^{(n)} >$ is bounded. Therefore, there is a subsequence $< x_{\alpha_{n_r}}^{(n_r)} >$ of $< x_{\alpha_n}^{(n)} >$ that converges to a fuzzy point $x_\alpha (\alpha \neq 0)$. Since $S(D)$ is closed and the elements of $< x_{\alpha_{n_r}}^{(n_r)} >$ belong to D, $x_\alpha \in D$. Hence f is fuzzy continuous at x_α , so that $< f(x_{\alpha_{n_r}}^{(n_r)}) >$ converges to $f(x_\alpha)$. The convergent fuzzy sequence $< f(x_{\alpha_{n_r}}^{(n_r)}) >$ must be bounded. But this is a contradiction. □

DEFINITION 2.7. Let D be a domain and let $f : D \to F_p(\mathbb{R})$. We say that f has a *maximum* on D. if there is a point $M_\lambda \in D$ such that $f(M_\lambda) \geq f(x_\alpha)$ for all $x_\alpha \in D$. We say that f has a *minimum* on D if there is a point $m_\mu \in D$ such that $f(m_\mu) \leq f(x_\alpha)$ for all $x_\alpha \in D$. M_λ is a *maximum fuzzy point* for f on D, and m_μ is a *minimum fuzzy point* for f on D if they exist.

THEOREM 2.3. *Let D be a fuzzy closed bounded domain and let $f : D \to F_p(\mathbb{R})$ be fuzzy continuous on D. Then f has a maximum and a minimum on D.*

Proof. Consider the non-empty set $f(D) = \{f(x_\alpha)|x_\alpha \in D\}$ of values of f on D. Then $f(D)$ is bounded. Let $(s^*)_\lambda = \sup f(D)$, $(s_*)_\mu = \inf f(D)$. We claim that there exist points M_α and m_β in D such that $(s^*)_\lambda = f(M_\alpha)$ and $(s_*)_\mu = f(m_\beta)$. Since $(s^*)_\lambda = \sup f(D)$, if $n \in \mathbb{N}$, then the point $(s^* - \frac{1}{n})_\lambda$ is not an upper bound of $f(D)$.

Consequently there exists a point $x_{\alpha_n}^{(n)} \in D$ such that

$$(s^* - \frac{1}{n})_\lambda \leq f(x_{\alpha_n}^{(n)}) \leq (s^*)_\lambda \quad \text{for} \quad n \in \mathbb{N}.$$

Since D is bounded, the sequence $X =< x_{\alpha_n}^{(n)} >$ is bounded. There is a subsequence $X' =< x_{\alpha_{n_r}}^{(n_r)} >$ of X that converges to some point M_α. Since D is a closed domain and the fuzzy points of X' belong to D. Then $M_\alpha \in D$. Therefore f is fuzzy continuous at M_α so that

$\lim < f(x^{(n_r)}_{\alpha_{n_r}}) >= f(M_\alpha)$. It follows that

$$(s^* - \frac{1}{n_r})_\lambda \leq f(x^{(n_r)}_{\alpha_{n_r}}) \leq (s^*)_\lambda \quad \text{for} \quad n \in \mathbb{N}.$$

We conclude that $\lim < f(x^{(n_r)}_{\alpha_{n_r}}) >= (s^*)_\lambda$. Therefore we have

$$f(M_\alpha) = \lim < f(x^{(n_r)}_{\alpha_{n_r}}) >= (s^*)_\lambda = \sup f(D).$$

We conclude that M_α is a maximum point of f on D. Similarly m_β is a minimum point of f on D. □

COROLLARY 2.4. *Let D be a fuzzy closed bounded domain and let $f : D \to F_p(\mathbb{R})$ be a fuzzy continuous function such that $S(f(x_\alpha)) > 0$ for each x_α in D. Then there exists a fuzzy point n_λ such that $f(x_\alpha) \geq n_\lambda$, $n > 0$ for all x_α in D.*

DEFINITION 2.8. Let D be a domain and $f : D \to F_P(\mathbb{R})$. We say that f is *uniformly fuzzy continuous* on D, if for each $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that if $x_\alpha, y_\beta \in D$ and if $|x - y| < \delta(\epsilon), |\alpha - \beta| < \delta(\epsilon)$ then $|S(f(x_\alpha)) - S(f(y_\beta))| < \epsilon, |V(f(x_\alpha)) - V(f(y_\beta))| < \epsilon$.

THEOREM 2.5. *Let D be a closed bounded domain and let $f : D \to F_p(\mathbb{R})$ be fuzzy continuous on D. Then f is uniformly fuzzy continuous on D.*

Proof. If f is not uniformly fuzzy continuous on D then there exists $\epsilon_0 > 0$ and two sequence $< x^{(n)}_{\alpha_n} >$ and $< y^{(n)}_{\beta_n} >$ in D such that $|x^{(n)} - y^{(n)}| < \frac{1}{n}$, $|\alpha_n - \beta_n| < \frac{1}{n}$ and $|S(f(x^{(n)}_{\alpha_n})) - S(f(y^{(n)}_{\beta_n}))| \geq \epsilon_0$, $|V(f(x^{(n)}_{\alpha_n})) - V(f(y^{(n)}_{\beta_n}))| \geq \epsilon_0$, for all $n \in \mathbb{N}$. Since D is bounded, the sequence $< x^{(n)}_{\alpha_n} >$ is bounded. Then there is a subsequence $< x^{(n_k)}_{\alpha_{n_k}} >$ of $< x^{(n)}_{\alpha_n} >$ that converges to an element x_α. Since D is closed, the limit x_α belongs to D. It is clear that the corresponding subsequence $< y^{(n_k)}_{\beta_{n_k}} >$ also converges to x_α, since

$$|y^{(n_k)} - x| \leq |y^{(n_k)} - x^{(n_k)}| + |x^{(n_k)} - x|.$$

Now if f is fuzzy continuous at x_α, then both of the sequences $< f(x_{\alpha_{n_k}}^{(n_k)}) >$ and $< f(y_{\beta_{n_k}}^{(n_k)}) >$ must converges to $f(x_\alpha)$. But this is not possible since

$$|S(f(x_{\alpha_n}^{(n)})) - S(f(y_{\beta_n}^{(n)}))| \geq \epsilon_0,$$

for all $n \in \mathbb{N}$. □

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