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ANOTHER NORMAL FUZZY R–SUBGROUPS IN NEAR–RINGS

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ABSTRACT. In this paper we discuss another normalization of fuzzy R-subgroups in near-rings.

1. Introduction

S. Abou-Zaid [1] introduced the notion of a fuzzy subnear-ring, and studied fuzzy left (resp. right) ideals of a near-ring, and many researchers [4, 5, 6, 12] are engaged in extending the concepts. K. H. Kim and Y. B. Jun [9] investigated fuzzification of R-subgroups in a near-ring R. The concept of normalization of fuzzy ideals was applied to BCK-algebras [5] and Gamma rings [3]. Recently, K. H. Kim and Y. B. Jun [10] introduced the notion of a normal fuzzy R-subgroup in a near-ring, and investigated some related properties. In this paper we discuss another normalization of fuzzy R-subgroups in near-rings.

By a *near-ring* [2] we mean a non-empty set R with two binary operations "+" and "·" satisfying the following axioms:

(i) $(R, +)$ is a group,

(ii) $(R, \cdot)$ is a semigroup,

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word "near-ring" in stead of "left

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near-ring". We denote xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. A *two sided* R*-subgroup* of a near-ring R is a subset H of R such that

(i) $(H, +)$ is a subgroup of $(R, +)$,

(ii) $RH \subset H$,

(iii) $HR \subset H$.

If H satisfies (i) and (ii) then it is called a *left* R*-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right* R*-subgroup* of R.

In what follows the letter R denotes a near-ring unless otherwise specified. Let μ be a fuzzy set in R. We say that μ is a *fuzzy subnear-ring* of R if, for all $x, y \in R$,

(F1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,

(F2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

If a fuzzy set μ in R satisfies the property (F1) then $\mu(0) \geq \mu(x)$ for all $x \in R$ (see [11, Lemma 2.3]).

2. Preliminaries

DEFINITION 2.1. ([1]) A fuzzy set μ in R is called a *fuzzy right (*resp. *left)* R*-subgroup* of R if

(F3) μ is a fuzzy subgroup of $(R, +)$,

(F4) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$.

Recall that a fuzzy set μ in R is a fuzzy right (resp. left) R-subgroup of R if and only if the level subset $\mu_t^{\geq}$ is a right (resp. left) R-subgroup of R, which is called a *level right* (resp. *left*) R*-subgroup* of R, where $t \in \mathrm{Im}(\mu)$ (see [9]).

EXAMPLE 2.2 ([9]). Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then $(R, +, \cdot)$ is a near-ring. We define a fuzzy subset $\mu : R \longrightarrow [0, 1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then μ is a fuzzy subgroup of $(R, +)$, and we have that $\mu(xr) \geq \mu(x)$ for all $r, x \in R$. Hence μ is a fuzzy right R-subgroup of R.

PROPOSITION 2.3. [10] *Let μ be a fuzzy right (resp. left) R-subgroup of a near-ring R, then the set $R_\mu := \{x \in R | \mu(x) = \mu(0)\}$ is a right (resp. left) R-subgroups of R.*

THEOREM 2.4. [10] *Let μ be a fuzzy left (right resp.) R-subgroup of R and let μ° be a fuzzy subset in R defined by*

$$\mu^\circ(x) := \frac{1}{\mu(0)}\mu(x)$$

for all $x \in R$. Then μ° is a normal fuzzy left (resp. right) ideal of R containing μ.

3. Main Results

THEOREM 3.1. *A fuzzy left (right resp.) R-subgroup μ of R is normal if and only if $\mu^\circ = \mu$.*

Proof. Assume that μ is a normal fuzzy left (right resp.) R-subgroup of R and let $x \in R$. Then $\mu^\circ(x) = \frac{1}{\mu(0)}\mu(x) = \mu(x)$, and hence $\mu^\circ = \mu$. □

THEOREM 3.2. *If μ is a fuzzy left (right resp.) R-subgroup of R, then* $(\mu^\circ)^\circ = \mu^\circ$.

Proof. For any $x \in R$ we have $(\mu^\circ)^\circ(x) = \frac{1}{\mu^\circ(0)}\mu^\circ(x) = \mu^\circ(x)$, since $\mu^\circ(0) = \frac{1}{\mu(0)}\mu(0) = 1$, completing the proof. □

Since μ is normal, $\mu^\circ = \mu$. By applying Theorem 3.1 we obtain:

COROLLARY 3.3. *If μ is a normal fuzzy left (right resp.) ideal of R, then* $(\mu^\circ)^\circ = \mu$.

THEOREM 3.4. *Let μ be a fuzzy left (right resp.) R-subgroup of R. If there exists a fuzzy left (right resp.) R-subgroup ν of R satisfying $\nu^\circ \subseteq \mu$, then μ is normal.*

Proof. Suppose there exists a fuzzy left (right resp.) R-subgroup ν of R such that $\nu^\circ \subseteq \mu$. Then $1 = \nu^\circ(0) \leq \mu(0)$, whence $\mu(0) = 1$. The proof is complete. □

By using Theorem 3.1, we have the following corollary.

COROLLARY 3.5. *Let μ be a fuzzy left (right resp.) R-subgroup of R. If there exists a fuzzy left (resp. right) R-subgroup ν of R satisfying $\nu^\circ \subseteq \mu$, then $\mu^\circ = \mu$.*

THEOREM 3.6. *Let μ be a non-constant normal fuzzy left (resp. right) R-subgroup of R, which is maximal in the poset of normal fuzzy left (right resp.) R-subgroups under set inclusion. Then μ takes only the values* 0 *and* 1.

Proof. Note that $\mu(0) = 1$. Let $x \in R$ be such that $\mu(x) \neq 1$. It is enough to show that $\mu(x) = 0$. Assume that there exists $a \in R$ such that $0 < \mu(a) < 1$. Define a fuzzy subset $\nu : R \to [0,1]$ by $\nu(x) := \frac{1}{2}\{\mu(x) + \mu(a)\}$ for all $x \in R$. Then clearly ν is well-defined.

Assume that μ is a normal fuzzy left R-subgroup of R. Let $x, y \in R$. Then

$$\begin{aligned}\nu(x-y) &= \frac{1}{2}\{\mu(x-y)+\mu(a)\}\\ &\geq \frac{1}{2}\{\min\{\mu(x),\mu(y)\}+\mu(a)\}\\ &= \min\{\frac{1}{2}\{\mu(x)+\mu(a)\},\frac{1}{2}\{\mu(y)+\mu(a)\}\}\\ &= \min\{\nu(x),\nu(y)\},\\ \nu(xr) &= \frac{1}{2}\{\mu(xr)+\mu(a)\}\\ &\geq \frac{1}{2}\{\mu(x)+\mu(a)\}\\ &= \nu(x)\end{aligned}$$

and

$$\begin{aligned}\nu(rx) &= \frac{1}{2}\{\mu(rx)+\mu(a)\}\\ &\geq \frac{1}{2}\{\mu(x)+\mu(a)\}\\ &= \nu(x).\end{aligned}$$

Hence ν is a fuzzy left R-subgroup of R. Now, we have

$$\begin{aligned}\nu^{\circ}(x) &= \frac{\nu(x)}{\nu(0)}\\ &= \frac{2}{\nu(0)+\nu(a)}\cdot\frac{\nu(x)+\nu(a)}{2}\\ &= \frac{\nu(x)+\nu(a)}{\nu(0)+\mu(a)}\end{aligned}$$

for all $x \in R$, and so $\nu^{\circ}(0) = 1$. Hence ν is a normal fuzzy right (resp. left) R-subgroup of R. Noticing that

$$\nu^{\circ}(0) = 1 > \nu^{\circ}(a) = \frac{2\mu(a)}{\mu(0)+\mu(a)} > \mu(a),$$

we know that ν° is non-constant. It follows from $\nu^{\circ}(a) > \mu(a)$ that μ is not maximal. This proves the theorem. □

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