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THE APPROXIMATE PERRON-STIELTJES INTEGRAL AND APPROXIMATE ROUSSEL DERIVATIVE

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ABSTRACT. In this paper, we introduce approximate Perron-Stieltjes integral and approximate Roussel derivative and investigate the differentiability of indefinite approximate Perron-Stieltjes integral.

1. Introduction

The Riemann-Stieltjes integral is useful in several areas of analysis as well as in probability theory and physics. It provides a connection between the integrals of Riemann and Lebesgue as well as a connection between integrals and infinite series.([1]) It was J. Radon who pointed out the importance of the Lebesgue-Stieltjes integral([4]) for certain classical parts of analysis, particularly for potential theory. The modern progress of this theory, which is bound up with the theory of subharmonic functions, has shown up still further the fruitfulness of the Lebesgue-Stieltjes integral in this branch of analysis. The Perron-Stieltjes integral([2],[3],[4],[5],[6]) is a generalization of the Perron integral and includes the Lebesgue-Stieltjes integral. In this paper, we introduce approximate Perron-Stieltjes integral and approximate Roussel derivative and investigate the differentiability of indefinite approximate Perron-Stieltjes integral

$$F(x) = (AP)\int_a^x f d\phi.$$

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2. The Definitions of the approximate Perron-Stieltjes Integral and the approximate Roussel Derivative

Among the various generalizations of the Stieltjes type for the Perron integral(for instance R. L. Jeffery, J. Ridder and A. J. Ward), that due to Ward has the advantage of including the others and of defining the process of Stieltjes integration with respect to any finite functions whatsoever. In what follows, an interval, by itself, always means either a closed non-degenerate interval or an empty set, unless another meaning is obvious from the context. A function of an interval $F(I)$ is said to be additive if $F(I_1 \cup I_2) = F(I_1) + F(I_2)$whenever I_1,$I_2$and $I_1 \cup I_2$ are intervals and I_1,I_2 are non-overlapping. If ϕ is a finite function defined on $[a, b]$, we define $\phi[E] = \{\phi(x) : x \in E\}$, and for $I = [c, d]$ we define $\phi(I) = \phi(d) - \phi(c)$. And I_x represents an interval containing x.
We define the density of a set at a point. This is a very important concepts in real analysis. The definition given below is not the most general definition of density, but is suitable for our purposes. We define $\mu^*$and μ represent the Lebesgue outer measure and Lebesgue measure respectively.

DEFINITION 2.1. Let E be a measurable set and let c be a real number. The density of E at c is defined by

$$d_c E = \lim_{h \to 0+} \frac{\mu(E \cap (c - h, c + h))}{2h}$$

provided the limit exists. It is clear that $0 \leq d_c E \leq 1$ when it exists. The point c is a point of density of E if $d_c E = 1$ and a point of dispersion of E if $d_c E = 0$.

It can be verified that almost all the points of a measurable set E are points of density of E and almost all the points of complement of E are points of dispersion of E . We need the following definitions.

DEFINITION 2.2. If E is a measurable set, then E^d represents the set of all points $x \in E$ such that $d_x E = 1$. Let S_x be a measurable set in $[a, b]$ with $x \in S_x^d$. We define $\triangle = \{S_x : x \in [a, b]\}$, which is called an approximate distribution on $[a, b]$.

REMARK 2.1. This $\triangle$ is not unique. We can take a different $\triangle$.

DEFINITION 2.3. Given two finite functions$f$ and ϕ on $[a, b]$, an additive interval function U will be termed a major function of f with respect to ϕ and $\triangle$ on $[a, b]$ if for some $\delta(x) > 0$, $U(I_x) \geq f(x)\phi(I_x)$ for every interval I_x whose endpoints are in $S_x \in \triangle$ with $\mu(I_x) < \delta(x)$. An additive interval function V will be termed a minor function of f with respect to ϕ and $\triangle$ on $[a, b]$if for some $\delta(x) > 0, V(I_x) \leq f(x)\phi(I_x)$ for every interval I_x whose endpoints are in $S_x \in \triangle$ with $\mu(I_x) < \delta(x)$.

DEFINITION 2.4. Let f and ϕ be finite functions on $[a, b]$. Then f is approximately Perron-Stieltjes(APS) integrable with respect to ϕ on $[a, b]$ if for some $\triangle$, f has at least one major function and one minor function of f with respect to ϕ and $\triangle$ on $[a, b]$ and the numbers
inf $\{U([a, b]) :$ $U$is a major function with respect to ϕ and $\triangle$ on $[a, b]\}$
sup $\{V([a, b]) : V$ is a minor function with respect to ϕ and $\triangle$ on $[a, b]\}$
are equal. This common value is the APS integral of f with respect to ϕ on $[a, b]$. We denote the integral as $\int_a^b f d\phi$ and use the prefix (AP) if it is necessary to distinguish this integral from others. The function f is APS integrable on a measurable set $E \subset [a, b]$ if $f\chi_E$ is APS integrable on $[a, b]$.

Then we have the following theorem.

THEOREM 2.1. *Let f be a finite function on $[a, b]$. Then f, is APS integrable with respect to ϕ on $[a, b]$ if and only if for each $\varepsilon > 0$ there exists a major function U and a minor function V of f with respect to ϕ and $\triangle$on $[a, b]$ such that*

$$U([a, b]) - V([a, b]) < \varepsilon$$

for some $\triangle$.

The APS integral has all of the usual properties of an integral. For example,
$(AP)\int_a^b(a_1f_1+a_2f_2)d(b_1\phi_1+b_2\phi_2)=\sum_{i,k=1,2}a_ib_k(AP)\int_a^b f_i d\phi_k$.

DEFINITION 2.5. A function $\phi:[a,b]\longrightarrow R$ satisfies condition(*) on $[a,b]$ if for every $x_0\in[a,b]$, ϕ is not constant on any interval containing x_0.

DEFINITION 2.6. Given two finite functions of a real variable F and ϕ, where ϕ satisfies condition(*). We shall say that a number L is the approximate Roussel derivative of F with respect to ϕ at a point x if for some $\triangle$,
(1)$\lim_{\mu(I_x)\to 0}[F(I_x)-L\cdot\phi(I_x)]=0$ and
(2)$\lim_{\mu(I_x)\to 0}\frac{|F(I_x)-L\cdot\phi(I_x)|}{\omega(\phi,I_x)}=0$
where the endpoints of I_x are in $S_x\in\triangle$ and $\omega(\phi,I_x)=sup\{|\phi(I)|:I\subset I_x\}$ which is called the oscillation of ϕ on I_x.We denote $L=\frac{rdF(x)}{rd\phi(x)}$.

LEMMA 2.2. ([4]) *Let ϕ be a finite function defined on $[a,b]$, E a bounded set in $[a,b]$, and T a system of intervals such that each point of E is an (right- or left-hand) end-point of an interval in T of arbitrarily small length. Then, for every number $M<\mu^*(\phi[E])$, we can select from T a finite system $\{I_k:1\le k\le q\}$ of non-overlapping intervals such that*

$$\sum_{k=1}^{q}\mu^*(\phi[I_k])\ge\frac{M}{2}.$$

3. The Main Result

For the Riemann-Stieltjes indefinite integral we have the following theorem: Let $f:[a,b]\to R$ be bounded and let $\phi:[a,b]\to R$ be of bounded variation on $[a,b]$. Suppose that f is Riemann-Stieltjes integrable on $[a,b]$ and define $F(x)=\int_a^x fd\phi$ for every $x\in[a,b]$. Then assuming that f is continuous on $[a,b]$,the function F is differentiable almost everywhere on $[a,b]$ and $F' = f\phi'$ almost everywhere on $[a,b]$. For the Roussel derivative, we have the following theorem.

THEOREM 3.1. *Let $f : [a,b] \to R$ be APS integrable with respect to ϕ on $[a,b]$, where ϕ satisfies condition(*), and let $F(x) = \int_a^x f d\phi$ for each $x \in [a,b]$. Then $\frac{rdF}{rd\phi} = f$ except at most those of a set E such that $\mu(\phi[E]) = 0$.*

Proof. Let $0 < \varepsilon \leq 1$. Then for some $\triangle$, there exist a major function U of f with respect to ϕ and $\triangle$ on $[a,b]$ such that $U([a,b]) - F([a,b]) < \varepsilon^2$. Let $H = U - F$ and let E_ε be the set of $x \in [a,b]$ which satisfies that for every $\delta > 0$ there exists an interval I_x with $\mu(I_x) < \delta$ such that $H(I_x) \geq \varepsilon\mu^*(\phi[I_x])$.
It follows that each point of E_ε is an endpoint of intervals I, as small as we please, which fulfils the inequality $H(I) \geq \frac{\varepsilon}{2}\mu^*(\phi[I])$. Suppose that $M < \mu^*(\phi[E_\varepsilon])$. By Lemma 2.8, there exists a finite system of non-overlapping intervals $\{I_k : 1 \leq k \leq q\}$ such that $H(I_k) \geq \frac{\varepsilon}{2} \cdot \mu^*(\phi[I_k])$ for $k = 1, \cdots, q$ and that $\displaystyle\sum_{k=1}^{q} \mu^*(\phi[I_k]) \geq \frac{M}{2}$. Consequently, since H is non-decreasing, $\varepsilon^2 > H([a,b]) \geq \frac{M\varepsilon}{4}$; and therefore $M < 4\varepsilon$, so $\mu^*(\phi[E_\varepsilon]) \leq 4\varepsilon$. Note that for every $x \notin E_\varepsilon$, there exists a number $\delta(x) > 0$ such that

$$H(I_x) \leq \varepsilon\mu^*(\phi[I_x])$$

for every I_x with $\mu(I_x) < \delta(x)$, and that $\mu^*(\phi[\bigcup_{k=1}^{\infty} E_{\frac{1}{k}}]) = 0$. Now let x be any point of $[a,b]$. Since U is a major function of f, for each $x \in [a,b]$ there exists a measurable set $S_x \in \triangle$ such that for some $\delta_1(x) > 0, U(I_x) \geq f(x)\phi(I_x)$ for every interval I_x whose endpoints are in S_x with $\mu(I_x) < \delta_1(x)$. Let

$$S = \bigcup_{x\in[a,b]} \{I_x : \text{ endpoints of } I_x \text{ are in } S_x\}.$$

We have for every interval $I_x \in S$ with $\mu(I_x) < \delta_1(x)$,
$F(I_x) - f(x)\phi(I_x) = U(I_x) - f(x)\phi(I_x) - H(I_x) \geq -H(I_x) > -\varepsilon^2$,
and if $x \notin E_\varepsilon$, there exists a positive number $\delta_2(x) < \delta_1(x)$ such that for every $I_x \in S$ with $\mu(I_x) < \delta_2(x)$,
$F(I_x) - f(x)\phi(I_x) \geq -H(I_x) \geq -\varepsilon \cdot \mu^*(\phi[I_x]) \geq -\varepsilon \cdot \omega(\phi, I_x)$.
Combining this with the similar upper evaluations of $F(I_x) - f(x)\phi(I_x)$ obtained by symmetry, we see, since ε is a arbitrary positive number, that $\frac{rdF(x)}{rd\phi(x)} = f(x)$ except at most those of a set $E = \bigcup_{k=1}^{\infty} E_{\frac{1}{k}}$ which satisfies that $\mu(\phi[E]) = 0$. □

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