JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 12, August 1999

ON SPECIAL DEFORMATIONS OF PLANE QUARTICS WITH AN ORDINARY CUSP OF MULTIPLICITY THREE

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ABSTRACT. Let $\{C_t\}$ be a pencil of smooth quartics for $t \neq 0$ degenerating to a plane quartic C_0 with an ordinary cusp of multiplicity 3. We compute the stable limit as $t \to 0$ of $\{C_t\}$ when the total surface of this family has a triple point at the singular point of C_0.

1. Introduction

Let $\{C_t\}$ be a pencil of curves where C_t are smooth curves of genus g for t in a punctured disk $\Delta^* = \Delta - 0 \subset \mathbb{C}$ and C_0 is a singular curve. Then there exists a morphism $\phi^* : \Delta^* \to \mathcal{M}_g$ which extends uniquely to $\phi : \Delta \to \overline{\mathcal{M}}_3$. $\phi(0)$ is called a stable limit of $\{C_t\}$ as $t \to 0$. It can be computed from (semi-)stable reduction theorem. Refer to the book [2] for stable reduction theorem. In this paper we study stable limits of $\{C_t\}$ for the pencils $\{C_t\}$ of plane quartics with C_t for $t \neq 0$ smooth quartics and C_0 a quartic with an ordinary cusp of multiplicity 3. In [3], we have showed that the stable limit as $t \to 0$ is smooth if the total surface of a family $\{C_t\}$ is smooth or has a double point at the singular point of the central fiber $C_0 = C$. From the direct computation, we show in section 2 that the stable limits of

*Supported in part by Chungnam National University in 1998
**Supported in part by Korea Research Foundation, Project No. 1998-015-D00010
Received by the editors on June 29, 1999.
1991 *Mathematics Subject Classifications*: 14H10.
Key words and phrases: family of curves, (semi-)stable reduction theorem, stable limits.

$\{C_t\}$ are same as the stable limits of reducible plane quartic $L+F$ where L is a line and F is a cubic with $I_{P'}(L,F)=3$ for some point P' with the total surface smooth at P' if the total surface $\{C_t\}$ has a triple point which is the only remaining case in [3].

2. The stable limits of the families that we study

We first introduce some etale versal deformation space. For further information, see [1]. Let $C=\{f(x,y)=0\}$ be a reduced curve with an isolated singular point $P=(0,0)$. Then there exists an etale versal deformation $\zeta:\mathcal{D}\to D$ defined by $D=\mathrm{Spec}(\mathbb{C}[x,y]/J)$ where J is the jacobian ideal of C generated by $(f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$ and $\mathcal{D}=\{f+\sum_1^N t_k h_k=0\}\subset \mathrm{Spec}(\mathbb{C}[x,y])\times\mathrm{Spec}(\mathbb{C}[a_1,a_2,\cdots,a_N])$ where $h_1,h_2,\cdots,h_N$ are basis of $\mathbb{C}[x,y]/J$. Then $\zeta:\mathcal{D}\to D$ becomes an etale versal deformation : versality means that any other deformation $\xi:\mathcal{X}\to X$ of C is analytically isomorphic to the pull back of $\zeta:\mathcal{D}\to D$. In this paper we call D the versal deformation space of C.

Let C be given by $y^3=x^4$ and $D=\mathrm{Spec}\mathbb{C}[x,y]/(y^2,x^3)$ the versal deformation space of C. Then $\dim_{\mathbb{C}}(D)=6$. Choose

$$\{1,x,y,x^2,xy,x^2y\}$$

as a basis of D and take (a,b,c,d,e,f) as coordinates of D. Then by the versality of D, every family $\{C_t\}$ degenerating to C as $t\to 0$ is defined by the equation

$$F(x,y,t)=y^3-x^4+\sum_{k\geq 1}t^k(a_k+b_kx+c_ky+d_kx^2+e_kxy+f_kx^2y)$$

which corresponds to a curve in D

$$\mathbf{r}(t)=\sum_{k=1}\mathbf{a}_kt^k$$

through the origin and which is smooth at the origin where $\mathbf{a}_k = (a_k, b_k, c_k, d_k, e_k, f_k) \in D$. Note that each fiber C_t can be projectified as a plane quartic in $\mathbb{P}^2$ by adding one smooth hyperflex point $(0:1:0)$. So it is a family of plane quartics degenerating to a plane quartic C with an ordinary cusp P of multiplicity 3. Since we concern the limits as $t \to 0$, we work over a small disk $\Delta \ni 0$.

Let $\mathcal{C}$ be a surface in $\mathbb{A}^2 \times \Delta$ (or in $\mathbb{P}^2 \times \Delta$ if one prefer) given by $F(x, y, t)$ with a projection $p : \mathcal{C} \to \Delta$ to the second component Δ. We always assume in this paper that $C_t = p^{-1}(t)$ is smooth for $t \neq 0$. So, P is the only singular point of $\mathcal{C}$. In [3] we have computed the corresponding stable limit when the total surface is smooth or has a double point at P.

In this section we assume that $\mathcal{C}$ has a triple point at the singular point P of the central fiber C. Then $\mathcal{C}$ is defined by

$$
\begin{aligned}
F(x,y,t) = {} & y^3 - x^4 + t(d_1x^2 + e_1xy + f_1x^2y) \\
& + t^2(b_2x + c_2y + d_2x^2 + e_2xy + f_2x^2y) \\
& + t^3(a_3 + b_3x + c_3y + d_3x^2 + e_3xy + f_3x^2y) \\
& + [t^4]
\end{aligned}
\tag{1}
$$

where $[t^4]$ means that all terms are the multiple of t^4. Since $\mathcal{C}$ is singular at P we first desingularize $\mathcal{C}$.

Let $\tilde{\pi} : \widetilde{\mathbb{A}}^3 \to \mathbb{A}^3$ be the blow-up of $\mathbb{A}^3$ at the origin and $\widetilde{\mathcal{C}}$ the proper transform of $\mathcal{C}$ under $\tilde{\pi}$. Put $\pi = \tilde{\pi}|\widetilde{\mathcal{C}} : \widetilde{\mathcal{C}} \to \mathcal{C}$. Then we have a new family $p_1 = p \circ \pi : \widetilde{\mathcal{C}} \to \Delta$ all fibers of which except $t = 0$ are same as C_t. $\widetilde{\mathbb{A}}^3$ is defined by $xy_1 = x_1y, xt_1 = x_1t, yt_1 = ty_1$ in $\mathbb{A}^3{}_{(x,y,t)} \times \mathbb{P}^2{}_{(x_1:y_1:t_1)}$. Then $\widetilde{\mathcal{C}}$ on each affine neighborhood U, V and W of $x_1 \neq 0$, $y_1 \neq 0$ and $t_1 \neq 0$ is given by the following equation respectively:

$$
(2)\qquad \begin{aligned}
\text{on } U_1: y_1^3 - x &+ t_1(d_1 + e_1y_1 + f_1xy_1)\\
&+ t_1^2(b_2 + c_2y_1 + d_2x + e_2xy_1 + f_2x^2y_1)\\
&+ t_1^3(a_3 + b_3x + c_3xy_1 + d_3x^2 + e_3x^2y_1 + f_3x^3y_1)\\
&+ [xt_1^4];
\end{aligned}
$$

$$
\begin{aligned}
\text{on } V_1: 1 - x_1^4y &+ t_1(d_1x_1^2 + e_1x_1 + f_1x_1^2y)\\
&+ t_1^2(b_2x_1 + c_2 + d_2x_1^2y + e_2x_1y + f_2x_1^2y^2)\\
&+ t_1^3(a_3 + b_3x_1y + c_3y + d_3x_1^2y^2 + e_3x_1y^2 + f_3x_1^2y^3)\\
&+ [yt_1^4];
\end{aligned}
$$

$$
(3)\qquad \begin{aligned}
\text{on } W_1: y_1^3 - x_1^4t &+ (d_1x_1^2 + e_1x_1y_1 + f_1x_1^2y_1t)\\
&+ (b_2x_1 + c_2y_1 + d_2x_1^2t + e_2x_1y_1t + f_2x_1^2y_1t^2)\\
&+ (a_3 + b_3x_1t + c_3y_1t + d_3x_1^2t^2 + e_3x_1y_1t^2 + f_3x_1^2y_1t^3)\\
&+ [t].
\end{aligned}
$$

LEMMA 1. *Under the assumption above, the new central fiber $p_1^*(0) = \widetilde{C}_0$ is a union of a rational curve $\bar{C}$ and a plane cubic F which meet at one point P_1 with $I_{P_1}(\bar{C}, F) = 3$. Moreover the total surface $\widetilde{\mathcal{C}}$ is smooth at P_1. Here $\bar{C}$ is the normalization of C and F the exceptional divisor of $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$.*

Proof. On U,

$$
\begin{aligned}
p_1^*(0) &= (t) = (t_1) + (x)\\
&= (t_1,\ y_1^3 - x)\\
&+ (x,\ y_1^3 + d_1t_1 + e_1y_1t_1 + b_2t_1^2 + c_2y_1t_1^2 + a_3t_1^3)\\
&= \bar{C} + F.
\end{aligned}
$$

Since $\{y_1, t_1\}$ is a local coordinates of C at P_1 the intersection point $I_{P_1}(\bar{C}, F)$ of $\bar{C}$ and F is equal to $I_P(t_1, y_1^3 + d_1t_1 + e_1y_1t_1 + b_2t_1^2 + c_2y_1t_1^2 + a_3t_1^3) = 3$. That P_1 is a smooth point of $\widetilde{C}$ follows from equation (2). □

LEMMA 2. *$\widetilde{C}_0$ is isomorphic to the reducible plane quartics $L + F$ with $I_{P_1}(L, F) = 3$ at some point P_1 where L is a line and F is a (possibly reducible and non-reduced) plane cubic.*

Proof. By Bezout's theorem, the total intersection number of a line and a cubic in $\mathbb{P}^2$ is 3. Now it follows from Lemma 1 and from that F is a plane cubic. □

The plane quartics $L+F$ with $I_{P_1}(L, F) = 3$ at some point P_1 where L is a line and F is a (possibly reducible and non-reduced) plane cubic has been studied in [4] when the family $\{C_t\}$ degenerating to $L + F$ is chosen generically, i.e., the total surface at the non-nodal singular point of $L+F$ is smooth, which is our case by lemma 1 if F is reduced except that F is an irreducible cubic with a cusp not at P_1. Note that all cases mentioned in the proof of Lemma 2 really happens. We are now ready to describe the stable limits of $\{C_t\}$ or $\{\widetilde{C}_t\}$ as $t \to 0$ when F is reduced.

THEOREM 1. *Suppose that F is reduced.*

(a) *If F is smooth at P_1, then the stable limit of $\{C_t\}$ as $t \to 0$ is either a genus two curve plus an elliptic curve which meet at one point or a genus two curve plus a rational curve with a node.*

(b) *If F has a node at P_1, then the stable limit of $\{C_t\}$ as $t \to 0$ is a genus 2 curve with one node.*

(c) *If F has a cusp at P_1, then the stable limit of $\{C_t\}$ as $t \to 0$ is a smooth curve of genus 3.*

(d) *If F has a triple point, then the stable limit of $\{C_t\}$ as $t \to 0$ becomes a smooth genus 3 curve.*

Note that all cases of Theorem 1 exist. In fact, we get (a) if $d_1 \neq 0$, (b) if $d_1 = 0, e_1 \neq 0$, (c) if $d_1 = e_1 = 0$, and (d) if $d_1 = e_1 = b_2 = 0$ in the equation (1).

Proof. Remember that $I_{P_1}(\bar{C}, F) = 3$ and $\widetilde{C}$ is smooth at P_1. If F is smooth at P_1, then it is isomorphic to either C5f, C6c or C7a in [4] according as F is irreducible, has a node, or has a cusp. So the result follows from theorem 3.2 in [4] except the last case. For the last case, all possible stable limit near cusp has been studied in [2]: the cusp part is replaced by an elliptic curve or a rational curve with one node. So the semi-stable limit of $\{C_t\}$ is a union of genus 2 curve and an elliptic curve or a rational curve with one node connected by the normalization of F. Since F is rational meeting other components at two points, it is contracted to give a stable curve of genus 3 which is a union of genus 2 curve and an elliptic curve or a rational curve with an node. If F has a node at P_1, it is isomorphic to either C6f, C6j in [4]. If F has a cusp at P_1, it is isomorphic to C7b in [4]. If F has a triple point at P_1, it is isomorphic to C8b in [4]. So all follow from Theorem 3.2 in [4] since $\widetilde{C}$ is smooth at only one non-nodal point P_1 of $\widetilde{C}_0$. □

If F is non-reduced, F is given by $y_1^3 + c_2 y_1 t_1^3 + a_3 t_1^3 = (y_1 - \gamma)^2 (y_1 + 2\gamma) = 0$ for some γ with its discriminant $27a_3^2 + 4c_2^3 = 0$. Then $\widetilde{C}$ has at best 4 double points of type A_1 if $\gamma \neq 0$ or type A_2 if $\gamma = 0$ which is the case (6) or (7) of theorem 4.2 in [4] when the multiple line of F and some quartic $g_1 = 0$ meet transversely as we write the equation (3) of $\widetilde{C}$

$$(y_1 - \gamma)^2 (y_1 + 2\gamma) + \sum_{k \geq 1} t^k g_k(x_1, y_1).$$

REMARK. For complete computation, all possible singular types of $\widetilde{\mathcal{C}}$ must be studied. They depend on the intersection types of the multiple line of F and the quartic $g_1 = 0$.

3. Families arising from the lines through the origin in the deformation space of $y^3 = x^4$

We now introduce some families of plane quartics degenerating to $y^3 = x^4$ given by a line in D through the origin. It together with remark in section 2 illustrates how complicate is the rational map from D to $\overline{\mathcal{M}}_3$. Now our family $\mathcal{C} = \{C_t\}$ is given by the equation $F(x,y,t) = y^3 - x^4 + t(a_1 + b_1x + c_1y + d_1x^2 + e_1xy + f_1x^2y)$. For C_t for $t \neq 0$ to be smooth, either a_1, b_1, or c_1 is not zero. So the stable limit of $\{C_t\}$ in this case is a smooth curve of genus 3. Now assume that $\{C_t\}$ is given by $F(x,y,t) = y^3 - x^4 + t(d_1x^2 + e_1xy + f_1x^2y)$. Then C_t for $t \neq 0$ has a node if $e_1 \neq 0$, has a cusp if $d_1 \neq 0,\ e_1 = 0$, or has a triple point if $d_1 = e_1 = 0$ with the total surface singular along $x = y = 0$ in all cases. It is the family of plane quartics with one node (or an ordinary cusp, or an ordinary triple point respectively) degenerating to a curve $y^3 = x^4$. To normalize $\mathcal{C}$, we blow up $\tilde{\pi} : \widetilde{\mathbb{A}^3} \to \mathbb{A}^3_{(x,y,t)}$ along the line $x = y = 0$. Let $\widetilde{\mathcal{C}}$ be the proper transform under $\tilde{\pi}$, $\pi = \tilde{\pi}|\widetilde{\mathcal{C}} : \widetilde{\mathcal{C}} \to \mathcal{C}$ and $p_1 : \widetilde{\mathcal{C}} \to \Delta$ be $p \circ \pi$. In $\widetilde{\mathcal{C}}$, we have normalized all singular points of C_t at the same time.

If C_t for $t \neq 0$ has a node, write $\mathcal{C}$ as $y^3 - x^4 + t\{x(d_1x + e_1y) + f_1x^2y\}$. Then $\widetilde{\mathcal{C}}$ is given by $y_1^3x - x^2 + t\{(d_1 + e_1y_1) + f_1xy_1\}$ in the affine neighborhood $x_1 \neq 0$ of $\widetilde{\mathbb{A}^3}$ which is given by $xy_1 = x_1y$ in $\mathbb{A}^3 \times \mathbb{P}^1_{[x_1:y_1]}$. So if $e_1 \neq 0$, we have a family of genus 2 curves degenerating to a reducible curve $\widetilde{C}_0$ consisting two rational components E and $\bar{C}$ which meet at some point P_1 with $I_{P_1}(E, \bar{C}) = 3$. Here E is the exceptional divisor of $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ and $\bar{C}$ the normalization of C. Note there exist two disjoint sections $s_1 : x_1 = 0$ and $s_2 : d_1x_1 + e_1y_1 = 0$ of $p_1 : \widetilde{\mathcal{C}} \to \Delta$

which is the pull back of singular locus of $\mathcal{C}$. So, these two sections meet E at two distinct points away from P_1. Now we take the usual stable reduction process while keeping these two sections. Then we get a new family $p' : \mathcal{C}' = \{C'_t\} \to \Delta$ with C'_t is isomorphic to $\widetilde{C_t}$ for $t \neq 0$ and C'_0 isomorphic to a reducible curve consisting of genus 2 curve meeting E at one point. Here two sections meet C'_0 at two points of E away from the intersection point. To get a family of stable curves of genus 3, we identify two sections. Therefore, the stable limit of $\{C_t\}$ is a genus 2 curve plus a rational curve with one node.

If C_t for $t \neq 0$ has a cusp, we may assume that $\mathcal{C}$ is given by $y^3 - x^4 + t(x^2 + f_1x^2y)$ and $\widetilde{\mathcal{C}}$ by $y_1^3x - x^2 + t(1 + f_1xy_1)$ with one section $s : x_1 = 0$ which is the pull back of singular locus of $\mathcal{C}$. So it is same as the case that C_t has a node except that we have one section. So, after the usual stable reduction process, we have a family of smooth genus two curves with one section which is obtained as the simultaneous normalization of cusps. So it is equivalent to finding a stable limit of genus 2 curve with one cusp which is either a genus 2 curve plus an elliptic curve or genus 2 curve plus a rational curve with one node.

If C_t for $t \neq 0$ has a triple point, then $\mathcal{C}$ is given by $y^3 - x^4 + tx^2y$ and $\widetilde{\mathcal{C}}$ by $y_1^3 - x + ty_1$. Now $\widetilde{\mathcal{C}}$ is a family of rational curves with three sections which meet at a point P_1 over $t = 0$. Note that three sections and the central fiber of $\widetilde{\mathcal{C}}$ have disjoint tangent lines. To separate these three sections we blow up $\widetilde{\mathcal{C}}$ at P_1. Then along the exceptional divisor, three sections and $\bar{C}$ are separated. After contracting $\bar{C}$, we get a family of rational curves with three disjoint sections. Now we identify all three sections to get a family of smooth rational curves with an ordinary triple point. So it is same as to find all possible stable limits of families of plane quartics degenerating to a quartic with an ordinary triple point.

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