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OSCILLATION OF NEUTRAL DIFFERENCE EQUATIONS

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ABSTRACT. We obtain some sufficient conditions for oscillation of the neutral difference equation with positive and negative coefficients

$$\Delta(x_n - cx_{n-m}) + px_{n-k} - qx_{n-l} = 0,$$

where Δ denotes the forward difference operator, m, k, l, are nonnegative integers, and $c \in [0,1), p, q \in \mathbb{R}^+$.

1. Introduction

The problem of oscillation and nonoscillation of solutions of neutral difference equations is receiving much attention by authors [2-6]. In particular Ladas [4] studied the difference equations with positive and negative coefficients

$$x_{n+1} - x_n + px_{n-k} - qx_{n-l} = 0, \quad n = 0, 1, 2, \cdots \tag{1}$$

$$x_{n+1} - x_n + px_{n+k} - qx_{n+l} = 0, \quad n = 0, 1, 2, \cdots, \tag{2}$$

where $p, q \in \mathbb{R}^+$ and $k, l \in \mathbb{N}$.

Now, we consider the neutral difference equations of the form

$$\Delta(x_n - cx_{n-m}) + px_{n-k} - qx_{n-l} = 0 \tag{3}$$

$$\Delta(x_n - cx_{n+m}) + px_{n+k} - qx_{n+l} = 0, \tag{4}$$

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where Δ denotes the forward operator : $\Delta x_n = x_{n+1} - x_n$ and c is a real number. If $c = 0$, then (3) and (4) reduce to (1) and (2), respectively. Let $l_0 = \max\{m, k, l\}$, where m, k, l are nonnegative integers. Then by a *solution* of (3) we mean a sequence $\{x_n\}$ which is defined for $n \geq N - l_0$ and satisfies (3) for $n \geq N$. A nontrivial solution $\{x_n\}$ of (3) is said to be *oscillatory* or *oscillates* if for every $N_1 \geq N$, there exists $n \geq N_1$ such that $x_n x_{n+1} \leq 0$. Otherwise, it is called *nonoscillatory*. Therefore a solution is *oscillatory* if it is eventually positive or eventually negative.

In this paper we obtain some sufficient conditions for the oscillation of solutions of equations (3) and (4).

2. Main Results

The following known lemmas will be used to prove the main theorems.

LEMMA 2.1. [5] *Consider the difference equation*

$$x_{n+1} - x_n + px_{n-k} = 0, \; n = 0, 1, 2, \cdots, \tag{5}$$

where $p \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Then every solution of equation (5) oscillates if and only if one of the following conditions holds :

(a) $k = -1$ and $p \leq -1$;

(b) $k = 0$ and $p \geq 1$;

(c) $k \in \{\cdots, -3, -2\} \cup \{1, 2, \cdots\}$ and $p\frac{(k+1)^{k+1}}{k^k} > 1$.

LEMMA 2.2. [3] *Assume that $p_i \in (0, \infty)$ and $q_i \in \mathbb{N}$ for $i = 1, 2, \cdots, m$. Then the following statements are true.*

(a) The difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} \leq 0, \; n = 0, 1, 2, \cdots$$

has an eventually positive solution if and only if the difference equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0, \ n = 0, 1, 2, \cdots$$

has an eventually positive solution.

(b) The difference inequality

$$x_{n+1} - x_n - \sum_{i=1}^{m} p_i x_{n+k_i} \geq 0, \ n = 0, 1, 2 \cdots$$

has an eventually positive solution if and only if the difference equation

$$y_{n+1} - y_n - \sum_{i=1}^{m} p_i y_{n+k_i} = 0, \ n = 0, 1, 2, \cdots$$

has an eventually positive solution.

LEMMA 2.3. [4] *Assume that*

$$p > q \geq 0, \ k \geq l \geq 0, \ q(k-l) \leq 1$$

and that

$$p - q > \frac{k^k}{(k+1)^{k+1}}, \qquad \text{if } k \geq 1$$

and $p - q \geq 1$ *if* $k = 0$.

Then every solution of equation (1) *oscillates.*

LEMMA 2.4. [4] *Assume that*

$$0 \leq p < q, \ 1 \leq k \leq l, \ p(l-k) \leq 1$$

and that

$$q - p > \frac{(l-1)^{l-1}}{l^l}, \qquad \text{if } l \geq 2$$

and $q - p \geq 1$ *if* $l = 0$.

Then every solution of equation (2) *oscillates.*

THEOREM 2.5. *Assume that*

$$p > q \geq 0, \ k \geq l \geq m \geq 0, \ q(k-l) + c \leq 1$$

and that

$$p - q > \frac{k^k}{(k+1)^{k+1}}, \quad \text{if } k \geq 1$$

and $p - q \geq 1$ *if* $k = 0$.

Then every solution of equation (3) oscillates.

Proof. For the case $k = l = m$ the equation (3) reduces to the following difference equation satisfying assumptions of Lemma 3 :

$$x_{n+1} - x_n + (p - q + c)x_{n-m} - cx_{n-(m-1)} = 0, \ n = 0, 1, 2, \cdots.$$

Thus every solution of equation (3) oscillates by Lemma 3. So suppose $k > l > m$. Assume that, for the sake of contradiction, equation (3) has an eventually positive solution $\{x_n\}$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq n_0$. Set

$$c_n = x_n - q \sum_{j=l+1}^{k} x_{n-j} - cx_{n-m}, \ n \geq n_0 + k. \tag{6}$$

Then we obtain

$$\begin{aligned} c_{n+1} - c_n &= x_{n+1} - x_n - q(x_{n-l} - x_{n-k}) - c_{n+1-m} + cx_{n-m} \\ &= -(p-q)x_{n-k} < 0, n \geq n_0 + k. \end{aligned} \tag{7}$$

Thus $\{c_n\}$ is a strictly decreasing sequence for $n \geq n_0 + k$. We claim that

$$\lim_{n \to \infty} c_n = L \in \mathbb{R}.$$

Otherwise, $L = -\infty$ and $\{x_n\}$ must be unbounded. Hence there exists $n_1 \geq n_0 + k$ such that $x_{n_1} = \max\{x_n \,:\, n \leq n_1\}$ and $c_n < 0$. Then

$$0 > c_{n_1} = x_{n_1} - q \sum_{j=l+1}^{k} x_{n_1 - j} - c x_{n_1 - m} \geq x_{n_1}[1 - q(k-l) - c] \geq 0,$$

which is a contradiction. Thus the limit L exists. It now follows, by taking the limit in (7), that $\lim_{n\to\infty} x_n = 0$. As the sequence $\{c_n\}$ decreases to zero, we conclude that

$$c_n > 0 \quad \text{for } n \geq n_0 + 2k. \tag{8}$$

Also, from (6) we see that $c_n < x_n$ for $n \geq n_0 + k$, so (7) yields the inequality

$$c_{n+1} - c_n + (p-q)c_{n-k} < 0 \quad \text{for } n \geq n_0 + 2k. \tag{9}$$

But in view of Lemmas 1,2 and hypothesis of Lemma 3, the difference inequality (9) can not have an eventually positive solution. This contradicts (8) and completes the proof of the theorem. □

THEOREM 2.6. *Assume that*

$$0 \leq p < q, \; 1 \leq m \leq k \leq l, \; p(l-k) + c \leq 1$$

and that

$$q - p > \frac{(l-1)^{l-1}}{l^l}, \quad \text{if } l \geq 2$$

and $q - p \geq 1$ *if* $l = 0$.

Then every solution of equation (4) oscillates.

Proof. For the case $m = k = l$ the equation (4) reduces to the following difference equation

$$x_{n+1} - x_n + (p - q + c)x_{n+k} - c x_{n+k+1} = 0.$$

Then every solution of (4) oscillates. So suppose $1 \le m < k < l$. Assume, for the sake of contradiction, that equation (4) has an eventually positive solution x_n. Set

$$c_n = x_n - p \sum_{j=n+k}^{n+l-1} x_j - c x_{n+m}. \tag{10}$$

Then we obtain

$$\begin{aligned} c_{n+1} - c_n &= x_{n+1} - x_n - p(x_{n+l} - x_{n+k}) - c(c_{n+1+m} - x_{n+m}) \\ &= (q-p)x_{n+l} > 0. \end{aligned} \tag{11}$$

Thus $\{c_n\}$ is eventually strictly increasing and either

$$\lim_{n\to\infty} c_n = \infty \tag{12}$$

or

$$\lim_{n\to\infty} c_n = c \in \mathbb{R}. \tag{13}$$

Assume that (13) holds. Then from (10) and (11) we see that

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} c_n.$$

Hence there exists an index n_1 such that

$$c_{n_1} < 0 \quad \text{and } x_{n_1} \ge x_n > 0 \quad \text{for } n \ge n_1.$$

Then (10) yields

$$\begin{aligned} 0 > c_{n_1} &= x_{n_1} - p \sum_{j=n_1+k}^{n_1+l-1} x_j - c x_{n_1+m} \\ &\ge x_{n_1}[1 - p(l-k) - c] \ge 0, \end{aligned}$$

which is a contradiction. Therefore (12) holds. From (10) and (11) we find

$$c_{n+1} - c_n - (p-q)c_{n+l} \geq 0,$$

and also $c_n > 0$. In view of Lemma 2 this implies that the difference equation

$$y_{n+1} - y_n - (p-q)y_{n+l} = 0$$

has an eventually positive solution. This contradicts, in view of Lemma 1, that

$$q - p > \frac{(l-1)^{l-1}}{l^l}, \quad \text{if } l \geq 2$$

and $q - p \geq 1$ if $l = 0$. This completes the proof of the theorem. □

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