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ON THE LIPSCHITZ CLASSES OF PERIODIC STOCHASTIC PROCESSES I

JONG MI CHOO

ABSTRACT. In this paper, T.Kawata's result[2] is generalized, by means of Hausdorff-Young inequality and Beurling's norm.

1. Introduction.

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and $X(t,\omega)$, $t \in \mathbf{R}$, is a stochastic process of the r-th order, $r \in [1, \infty)$, that is,

$$||X(t,\omega)||_r = (E|X(t,\omega)|^r)^{\frac{1}{r}} < \infty.$$

We say $X(t,\omega)$ is periodic with period 2π, if

$$||X(t+2\pi,\omega) - X(t,\omega)||_r = 0 \qquad \text{for every } t$$

and

$$\int_{-\pi}^{\pi} ||X(t,\omega)||_r^r \, dt < \infty.$$

The class of 2π-periodic processes of the r-th order will be denoted by L_p^r.

Let $\phi(h)$ be a positive nondecreasing function of h$\in$(0,1]. Write

$$\Delta_h^j X(t,\omega) = \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} X(t+\nu h, \omega),$$

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where j is a positive integer. The class of $X(t,\omega)$ with the property

$$\sup_{h>0}\int_{-\pi}^{\pi}\left(\frac{||\Delta_h^j X(t,\omega)||_r}{\phi(h)}\right)^r dt < \infty$$

is denoted by $\Delta^*_{j,r}(\phi)$ and is called the Lipschitz class $\Delta^*_{j,r}(\phi)$.

For a stochastic process $X(t,\omega)$ of L_p^r which belong to $\Delta^*_{j,r}(\phi)$, we consider the Fourier series

$$X(t,\omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega)e^{int},$$

where

$$C_n(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(t,\omega)e^{-int}\,dt.$$

T.Kawata[2] proved the following theorem 1 relating to discuss sample continuity and differentiability of stochastic processes.

THEOREM 1. *Let* $X(t,\omega) \in L_p^a, 1 < a \leq 2$ *and* $\frac{1}{a} + \frac{1}{a'} = 1$. *Let* K *be a given nonnegative integer. If there exists a positive integer* j *such that*

$$\sum_{n=1}^{\infty} n^{k-\frac{1}{a'}}[\phi(\frac{1}{n})]^{-1}M_a^{(j)}(\frac{1}{n}) < \infty,$$

then

$$\sum_{n=-\infty}^{\infty} |n|^k[\phi(\frac{1}{n})]^{-1}|C_n(\omega)| < \infty$$

almost surely, where ϕ *is a positive nondecreasing continuous function on (0,1] such that* $\phi(2t) < K\phi(t)$ *and*

$$M_a^{(j)}(\delta) = \sup_{|h|\leq\delta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}||\Delta_h^j X(t,\omega)||_a^a\,dt\right)^{\frac{1}{a}}.$$

In this paper, we extend the above result by the Hausdorff-Young inequality and the Beurling norm. (K is a constant which may be different in each occurrence.)

2. Main Results.

THEOREM 2. *Let* $1 < a \le 2, \dfrac{1}{a} + \dfrac{1}{a'} = 1, 0 < p \le a'$ *and* j *be a positive integer. Then, for* $X(t,\omega) \in L_p^a$, *we have*

$$
{}_{a'}\|||C_n(\omega)||_a |n|^{\frac{1}{a'}-\frac{1}{p}}[I(\frac{1}{|n|})]^{-1}||_p
\le K\left(\int_0^1 \left[\int_{-\pi}^{\pi} ||\Delta_h^j X(t,\omega)||_a^a \, dt\right]^{\frac{p}{a}} [I(h)^p h]^{-1} \, dh\right)^{\frac{1}{p}},
$$

where we denote $I(h)$ *a positive nondecreasing function on (0,1] such that* $I(2h) \le KI(h)$ *and* ${}_a||\cdot||_p$ *is the Beurling norm.*

Theorem 2 is an analogue of a result in [4] for the ordinary Fourier series. The proof of Theorem 2 is based on the following inequality.

THEOREM 3. (I) *Let* $1 \le a \le 2$ *and let* a' *be the conjugate exponent of* a. *Let* $X(t,\omega) \in L_p^a$, *then*

$$
\left[\sum_{n=-\infty}^{\infty} ||C_n(\omega)||_a^{a'}\right]^{\frac{1}{a'}} \le K\left[\int_{-\pi}^{\pi} ||X(t,\omega)||_a^a \, dt\right]^{\frac{1}{a}}.
$$

(II) *Let* $1 \le a \le 2$ *and let* a' *be the conjugate exponent of* a. *If* $\{||C_n(\omega)||_a\} \in l^a$, *then there exists a stochastic process* $X(t,\omega) \in L_p^{a'}$ *such that* $C_n(\omega)$ *is the Fourier coefficients of* $X(t,\omega)$ *and*

$$
\left[\int_{-\pi}^{\pi} ||X(t,\omega)||_a^{a'} \, dt\right]^{\frac{1}{a'}} \le K\left[\sum_{n=-\infty}^{\infty} ||C_n(\omega)||_a^a\right]^{\frac{1}{a}}.
$$

Proof. (I) By the Hausdorff - Young inequality for the ordinary Fourier series, we have

$$
\left[\sum_{n=-\infty}^{\infty} |C_n(\omega)|^{a'}\right]^{\frac{a}{a'}} \le K\int_{-\pi}^{\pi} |X(t,\omega)|^a \, dt.
$$

Taking expectations of both sides, we have

$$(2) \qquad E\left[\sum_{n=-\infty}^{\infty}|C_n(\omega)|^{a'}\right]^{\frac{a}{a'}} \leq K\int_{-\pi}^{\pi}||X(t,\omega)||_a^a\,dt.$$

Now we estimate the left from below. Let $\frac{a'}{a}=\alpha\geq 1$. Then, by the Minkowski inequality,

$$(3) \qquad \left[\sum_{n=-\infty}^{\infty}(E|C_n(\omega)|^a)^\alpha\right]^{\frac{1}{\alpha}} \leq E\left[\sum_{n=-\infty}^{\infty}|C_n(\omega)|^{a\alpha}\right]^{\frac{1}{\alpha}} = E\left[\sum_{n=-\infty}^{\infty}|C_n(\omega)|^{a'}\right]^{\frac{a}{a'}}$$

The left of (3) is $\left[\sum_{n=-\infty}^{\infty}||C_n(\omega)||_a^{a'}\right]^{\frac{a}{a'}}$ and the last part of(3)is the left of (2). Therefore we have the conclusion.

(II) By the assumption, we have $\{C_n(\omega)\}\in l^a$ almost surely. Therefore, by the classical discussion, we have a stochastic process $X(t,\omega)$ whose Fourier coefficients is $C_n(\omega)$ and

$$\left[\int_{-\pi}^{\pi}|X(t,\omega)|^{a'}\,dt\right]^{\frac{a}{a'}} \leq \sum_{n=-\infty}^{\infty}|C_n(\omega)|^a.$$

Taking expectations of both sides and using the Minkowski inequality, we have, for $\alpha=\frac{a'}{a}$,

$$\left(\int_{-\pi}^{\pi}[E|X(t,\omega)|^a]^\alpha\,dt\right)^{\frac{1}{\alpha}} \leq E\left[\int_{-\pi}^{\pi}|X(t,\omega)|^{a\alpha}\,dt\right]^{\frac{1}{\alpha}} = E\left[\int_{-\pi}^{\pi}|X(t,\omega)|^{a'}\,dt\right]^{\frac{a}{a'}} \leq \sum_{n=-\infty}^{\infty}E|C_n(\omega)|^a,$$

that is,

$$\left[\int_{-\pi}^{\pi} ||X(t,\omega)||_a^{a'}\, dt\right]^{\frac{a}{a'}} \le \sum_{n=-\infty}^{\infty} ||C_n(\omega)||_a^a,$$

which is the required. □

Once we get the above inequality, the proof of Theorem 2 is a repetition of the argument in the ordinary Fourier series[4].

Now we can derive Theorem 1 from Theorem 2. We need to discuss the case $p = 1$, since $||C_n||_1 \le_a ||C_n||_1$ if $a \ge 1$. Take $I(h) = \phi(h)h^\delta$, where $\delta = k + 1/a$. Then, the inner part of the left side norm of the inequality in (1) is

$$||C_n(\omega)||_a |n|^{\frac{1}{a'}-1+\delta}[\phi(\frac{1}{|n|})]^{-1} = ||C_n(\omega)||_a |n|^k [\phi(\frac{1}{|n|})]^{-1}$$

which is not less than $E|C_n(\omega)||n|^k[\phi(\frac{1}{|n|})]^{-1}$.

On the other hand, the right side is not greater than

$$\begin{aligned} K\int_0^1 & h^{-1-\delta}[\phi(h)]^{-1} M_a^{(j)}(h)\, dh \\ & \le K\sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} h^{-1-\delta}[\phi(h)]^{-1} M_a^{(j)}(h)\, dh \\ & \le K\sum_{n=1}^{\infty} n^{1+\delta-2}[\phi(\frac{1}{n+1})]^{-1} M_a^{(j)}(\frac{1}{n}). \end{aligned}$$

Since $1+\delta-2 = k - 1/a'$ and $\phi(\frac{1}{n+1}) \ge K\phi(\frac{1}{n})$, we have the desired result. □

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Department of Mathematics
Mokwon University
Taejon 302-729, Korea

E-mail: jmchoo@mokwon.ac.kr