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GENERALIZED REGULAR HOMOMORPHISM

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ABSTRACT. In this paper, we introduce a generalized regular homomorphism as the extending notion of the regular homomorphism.

In [1], J. Auslander introduced regular minimal sets which are the universal minimal sets for certain properties. The class of regular minimal sets is shown to coincide with the minimal right ideals of enveloping semigroups of transformation groups. The classification of minimal sets has been strengthened and extended by consideration of homomorphisms.

In this paper, we introduce a generalized regular homomorphism, in connection with a certain subset of the automorphism group $\mathcal{G}$ of the universal minimal set (M, T), as the expending notion of the regular homomorphism defined by P. Shoenfeld ([6]).

Throughout this paper, (X, T) will denote the transformation group with compact Hausdorff phase space X. A closed nonempty subset A of X is called a *minimal subset* if for every $x \in A$, the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point*. If X is itself minimal, we say that it is a *minimal set*.

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points x and y of X are called *proximal* provided that for each index α of X,

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there exists a $t \in T$ such that $(xt, yt) \in \alpha$. The set of all proximal pairs is denoted by $P(X)$.

A continuous map π from (X,T) to (Y,T) with $\pi(xt) = \pi(x)t \quad (x \in X)$ is called a *homomorphism*. If Y is minimal, π is always onto. Especially, if π is onto, π is called an *epimorphism*. A homomorphism π from (X,T) onto itself is called an *endomorphism* of (X,T), and an isomorphism $\pi : (X,T) \to (X,T)$ is called an *automorphism* of (X,T). We denote the group of automorphisms of X by $A(X)$.

The *enveloping semigroup* of (X,T) denoted $E(X)$, or E is defined to be the closure T in X^X, providing with its product topology. The enveloping semigroup $E(X)$ is thus a compact semigroup in X^X. The *minimal right ideal* I is the nonempty subset of $E(X)$ with $IE \subset I$, which contains no proper nonempty subset of the same property.

DEFINITION 1. Let (X,T) be a transformation group. Two points x and x' are said to be *regular* if $h(x)$ and x' are proximal for some automorphism h of X. The set of all regular pairs in X is denoted by $R(X)$.

DEFINITION 2. Let T be an arbitrary topological group. A minimal transformation group (M,T) is said to be *universal* if every minimal transformation group with acting group T is a homomorphic image of (M,T). The group of automorphisms of (M,T) is denoted by $\mathcal{G}$. Given a minimal transformation group (X,T) and a homomorphism $\gamma : M \to X$, we define $G(X,\gamma) = \{\alpha \in \mathcal{G} \mid \gamma\alpha = \gamma\}$.

DEFINITION 3. ([6]) Let (X,T) and (Y,T) be minimal transformation groups and let (M,T) be a universal minimal set. A homomorphism $\pi : X \to Y$ is said to be *regular* if for given $\gamma : M \to X$ and $\delta : M \to X$ with $\pi\delta = \pi\gamma$, there exists an automorphism $h : X \to X$ such that $h\delta = \gamma$ and $\pi h = \pi$.

LEMMA 1. *Let (M,T) be a universal minimal set, (X,T) a minimal set, and let $\gamma : M \to X$ be a homomorphism. Let (x,x') be an almost* *periodic point of* $(X \times X, T)$. *Then there exists an* $m \in M$ *and an automorphism* $\alpha \in \mathcal{G}$ *such that* $\gamma(m) = x$ *and* $\gamma\alpha(m) = x'$.

Proof. Let (x, x') be an almost periodic point of $(X \times X, T)$. Then there exists an almost periodic point $(m, m') \in M \times M$ such that $\gamma^*(m, m') = (x, x')$, where $\gamma^* : M \times M \to X \times X$ is defined by $\gamma^*(m, m') = (\gamma(m), \gamma(m'))$. Since M is regular minimal, given m and m', there is an $\alpha \in \mathcal{G}$ such that

$$\alpha(m) = m' .$$

Therefore, $\gamma(m) = x$ and $\gamma\alpha(m) = \gamma(m') = x'$. □

Let (M, T) be a universal minimal set and let (X, T) be a minimal transformation group. Given a homomorphism $\gamma : M \to X$ and an automorphism $h : X \to X$, let

$$S_h(X, \gamma) = \{\alpha \in \mathcal{G} \mid h\gamma\alpha = \gamma \}.$$

It is clear that if we take $h = 1_X$, then $S_h(X, \gamma)$ coincides with $G(X, \gamma)$. For a homomorphism $\pi : X \to Y$ and an automorphism $k : Y \to Y$, $S_k(Y, \pi\gamma)$ is defined obviously.

Remark 1. $S_h(X, \gamma)$ is nonvoid for each $h \in A(X)$. In fact, let $x \in X$. Then $(h(x), x) \in X \times X$ is an almost periodic point. Therefore, there exist an $m \in M$ and an $\alpha \in \mathcal{G}$ such that $\gamma(m) = h(x)$ and $\gamma\alpha(m) = x$ by Lemma 4. Thus,

$$h\gamma\alpha(m) = h(x) = \gamma(m),$$

which implies that $\alpha \in S_h(X, \gamma)$.

For a given $h \in A(X)$, let us set

$$R_h(X) = \{(x, x') \in X \times X \mid (x, h(x')) \in P(X) \}.$$

Then the set of all regular pairs $R(X)$ can be written as the union of all $R_h(X)$ $(h \in A(X))$.

REMARK 2. Let (X,T) be a minimal set and let (x,x') be an almost periodic point of $(X \times X, T)$. Then $(x,x') \in R_h(X)$ if and only if $h(x') = x$.

A subset $S_h(X,\gamma)$ of $\mathcal{G}$ plays an important role to define a generalized regular homomorphism. Let $\alpha \in S_h(X,\gamma)$ for some $\alpha \in \mathcal{G}$ and let $m \in M$. Then $(m, \alpha(m))$ is an almost periodic point of $(M \times M, T)$. Define $\gamma(m) = x$ and $\gamma\alpha(m) = x'$. Then (x,x') is also an almost periodic point of $(X \times X, T)$. Since $\alpha \in S_h(X,\gamma)$, we obtain that

$$h(x') = h\gamma\alpha(m) = \gamma(m) = x \ .$$

We conclude that $\alpha \in S_h(X,\gamma)$ implies that there exist x and x' such that (x,x') is almost periodic and $(x,x') \in R_h(X)$.
Now, conversely, suppose that there exist x and x' in X such that $(x,x') \in X \times X$ is almost periodic and $(x,x') \in R_h(X)$. Then by Remark 2, $h(x') = x$. Since (x,x') is almost periodic, there exist $m \in M$ and $\alpha \in \mathcal{G}$ such that $\gamma(m) = x$ and $\gamma\alpha(m) = x'$ by Lemma 4. Therefore,

$$h\gamma\alpha(m) = h(x') = x = \gamma(m) \ ,$$

that is, $h\gamma\alpha = \gamma$ and that $\alpha \in S_h(X,\gamma)$.

The next theorem follows from the discussion in the previous paragraph.

THEOREM 1. *Let (X,T) be a minimal transformation group, (M,T) a universal minimal set, $h \in A(X)$ and $\gamma : M \to X$ be a homomorphism. Then the following are equivalent ;*

(1) $\alpha \in S_h(X,\gamma)$ *for some* $\alpha \in \mathcal{G}$.
(2) *There exist x and x' such that (x,x') is an almost periodic point of $(X \times X, T)$ and $(x,x') \in R_h(X)$.*

The following theorem is analogous to Theorem 1 ([2]).

THEOREM 2. *Let (X,T) and (Y,T) be minimal transformation groups, (M,T) a universal minimal set and let $\pi : X \to Y$, $\gamma : M \to X$ be homomorphisms. The following are equivalent ;*

(1) *There exists $x,\ x' \in X$ with $(\pi(x),\pi(x')) \in R(Y)$ such that $(x,x') \notin R(X)$.*

(2) *$\alpha \in S_k(Y,\pi\gamma) - S_h(X,\gamma)$ for some $k \in A(Y)$ and for all $h \in A(X)$.*

Proof. (1) $\Rightarrow$ (2) : Let $x,\ x' \in X$ with $(\pi(x),\pi(x')) \in R(Y)$. Then

$$\pi(x)q = k\pi(x')q$$

for all q in a minimal right ideal J in $E(Y)$ and for some $k \in A(Y)$. Let v be an idempotent of J. Then $\bar{\pi}(u) = v$ for some idempotent u in $E(X)$, where $\bar{\pi} : (E(X),T) \to (E(Y),T)$ is the epimorphism induced by π. It follows that

$$\pi(xu) = \pi(x)\bar{\pi}(u) = \pi(x)v = k\pi(x')v = k\pi(x')\bar{\pi}(u) = k\pi(x'u).$$

Since $(xu,x'u) = (xu,x'u)u$, $(xu,x'u)$ is an almost periodic point of $(X \times X,T)$. By Lemma 4, there exist $m \in M$ and $\alpha \in \mathcal{G}$ such that

$$\gamma(m) = xu \quad \text{and} \quad \gamma\alpha(m) = x'u.$$

It follows that

$$k\pi\gamma\alpha(m) = k\pi(x'u) = k\pi(x')v = \pi(x)v = \pi(xu) = \pi\gamma(m)$$

and thus $\alpha \in S_k(Y,\pi\gamma)$. Furthermore, $(xu,x'u) \notin R(X)$, because $(x,x') \notin R(X)$. Thus $h(xu) \neq x'u$ for all $h \in A(X)$, that is, $h\gamma\alpha(m) \neq \gamma(m)$, which shows that $\alpha \notin S_h(X,\gamma)$.

(2) $\Rightarrow$ (1) : Let $m \in M$ and let $\alpha \in S_k(Y,\pi\gamma) - S_h(X,\gamma)$ for some $k \in A(Y)$ and for all $h \in A(X)$. Then $(m,\alpha(m))$ is an almost periodic point of $(M \times M,T)$. If we put $x = \gamma(m)$, $x' = \gamma\alpha(m)$, then (x,x') is an almost periodic point of $(X \times X,T)$. Since $\alpha \in S_k(Y,\pi\gamma) - S_h(X,\gamma)$,

$$\pi(x) = \pi\gamma(m) = k\pi\gamma\alpha(m) = k\pi(x')$$

for some $k \in A(Y)$, but

$$h(x') = h\gamma\alpha(m) \neq \gamma(m) = x$$

for all $h \in A(X)$. Therefore, by Remark 6, $(\pi(x), \pi(x')) \in R(Y)$, but $(x, x') \notin R(X)$. □

DEFINITION 4. Let (X, T) and (Y, T) be minimal transformation groups, (M, T) a universal minimal set and let $k \in A(Y)$. A homomorphism $\pi : X \to Y$ is said to be *k-regular* if for given $\gamma : M \to X$, $\delta : M \to X$ with $k\pi\delta = \pi\gamma$, there exists an $h \in A(X)$ such that $h\delta = \gamma$ and $k\pi = \pi h$.

REMARK 3. If we take $k = 1_Y$, the identity of $A(Y)$, then k-regular homomorphism coincides with regular homomorphism.

LEMMA 2. *([2]) Let (X, T) be a minimal transformation group, (M, T) a universal minimal set and let $\gamma : M \to X$ and $\delta : M \to X$ be homomorphisms. Then there exists an $\alpha \in G$ such that $\delta = \gamma\alpha$.*

THEOREM 3. *Let (X, T) and (Y, T) be minimal transformation groups, (M, T) a universal minimal set $k \in A(Y)$ and let $\pi : X \to Y$, $\gamma : M \to X$ be homomorphisms. The following are equivalent ;*

(1) *$S_k(Y, \pi\gamma) \subset S_h(X, \gamma)$ for some $h \in A(X)$.*
(2) *For any $x, x' \in X$ with (x, x') almost periodic and $(\pi(x), \pi(x')) \in R_k(Y)$, there exists an $h \in A(X)$ such that*

$$h(x') = x \text{ and } \pi h = k\pi.$$

Proof. (1) ⇒ (2) : Suppose that $S_k(Y, \pi\gamma) \subset S_h(X, \gamma)$ for some $h \in A(X)$. Let $x, x' \in X$ with (x, x') almost periodic and $(\pi(x), \pi(x')) \in R_k(Y)$. Since $(x.x')$ is almost periodic, there exist $m \in M$ and $\alpha \in \mathcal{G}$ such that $\gamma(m) = x$, $\gamma\alpha(m) = x'$ by Lemma 4. Since $(\pi(x), \pi(x'))$ is almost periodic and $(\pi(x), \pi(x')) \in R_k(Y)$. From Remark 6, we obtain

$$k\pi(x') = \pi(x),$$

and

$$k\pi\gamma\alpha(m) = k\pi(x') = \pi(x) = \pi\gamma(m).$$

This shows that $\alpha \in S_k(Y, \pi\gamma)$, and by assumption $\alpha \in S_h(X, \gamma)$ for some $h \in A(X)$. It follows that

$$h(x') = h\gamma\alpha(m) = \gamma(m) = x$$

and

$$\pi h(x') = \pi(x) = k\pi(x').$$

that is, $h(x') = x$ and $\pi h = k\pi$.

(2) $\Rightarrow$ (1) : Let $\alpha \in S_k(Y, \pi\gamma)$. Then by Theorem 7, there exist y, y' in Y, such that (y, y') is almost periodic and $(y, y') \in R_k(Y)$. Since (y, y') is an almost periodic point of $(Y \times Y, T)$, there exists almost periodic point $(x, x') \in X \times X$ such that $\breve{\pi}(x, x') = (\pi(x), \pi(x')) = (y, y')$, where $\breve{\pi} : X \times X \to Y \times Y$ is the homomorphism induced by $\pi : X \to Y$. Therefore, by assumption, there exists an $h \in A(X)$ such that $h(x') = x$ and $\pi h = k\pi$. Since (x, x') is an almost periodic point, there is an (m, m') in $M \times M$ such that $\gamma(m) = x$ and $\gamma(m') = x'$. Define $\alpha(m) = m'$. Then,

$$\gamma(m) = x = h(x') = h\gamma(m') = h\gamma\alpha(m),$$

we conclude that $h\gamma\alpha = \gamma$ and thus $\alpha \in S_h(X, \gamma)$. □

THEOREM 4. *Let (X, T) and (Y, T) be minimal transformation groups, (M, T) a universal minimal set $k \in A(Y)$ and let $\pi : X \to Y$, $\gamma : M \to X$ be homomorphisms. The following are equivalent ;*

(1) *π is k-regular.*

(2) *$S_k(Y, \pi\gamma) \subset S_h(X, \gamma)$ for some $h \in A(X)$.*

Proof. (1) $\Rightarrow$ (2) : Let $\alpha \in S_k(Y, \pi\gamma)$. Then $k\pi\gamma\alpha = \pi\gamma$. Put $\delta = \gamma\alpha$. Since π is k-regular, there is an $h \in A(X)$ such that $h\gamma\alpha = h\delta = \gamma$, and $\pi h = k\pi$. This show that $\alpha \in S_h(X, \gamma)$.

(2) $\Rightarrow$ (1) : Let γ, δ be given with $k\pi\delta = \pi\gamma$. By Lemma 11, there is an $\alpha \in G$ such that $\delta = \gamma\alpha$, and so $k\pi\gamma\alpha = k\pi\delta = \pi\gamma$, that

is $\alpha \in S_k(Y, \pi\gamma)$. By assumption, we have $\alpha \in S_h(X, \gamma)$ for some $h \in A(X)$, and therefore,

$$h\delta = h\gamma\alpha = \gamma.$$

Now, let $m \in M$. If we let $x = \gamma\alpha(m)$ and $x' = \gamma(m)$, then

$$k\pi(x) = k\pi\gamma\alpha(m) = \pi\gamma(m) = \pi(x') = \pi h(x),$$

which shows that

$$k\pi = \pi h.$$

Therefore, π is k-regular. □

Proposition 2.2.8([6]) is the corollary of Theorem 3 and Theorem 4.

COROLLARY 5. *Let (X,T) and (Y,T) be minimal transformation groups, (M,T) a universal minimal set and let $\pi : X \to Y$, $\gamma : M \to X$, $\delta : M \to X$ be homomorphisms. Then the following are equivalent ;*

(1) *π is regular.*
(2) *$G(Y, \pi\gamma) \subset S_h(X, \gamma)$ for some $h \in A(X)$.*
(3) *For any two points $x, x' \in X$ with $\pi(x) = \pi(x')$, there exists an $h \in A(X)$ such that $(x, x') \in R_h(X)$ and $\pi h = \pi$.*
(4) *For any two points $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$, there exists an $h \in A(X)$ such that $h(x') = x$ and $\pi h = \pi$.*

REFERENCES

1. J. Auslander, Regular minimal sets I, Trans. Amer. Math. Soc. 123(1966), 469-479.
2. ________, Homomorphisms of minimal transformation group, Topology 9 (1970), 195-203.
3. R. Ellis, A semigroup associated with a transformation group, Trans. Amer. Math. Soc. 94(1960), 272-281.
4. ________, Universal minimal sets, Proc. Amer. Math. Soc 11(1960).
5. ________, Lectures on topological dynamics, W. A. Benjamin, Inc, 1969.

6. P. Shoenfeld, Regular homomorphisms of minimal sets, Doctoral Dissertation, University of Maryland, 1974.
7. M. H. Woo, Regular transformation groups, J. Korean Math. Soc. vol. 15 (1979), 129-137.
8. J. O. Yu, Regular relations in transformation groups, J. Korean Math. Soc. vol. 21 (1984), 41-48.

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