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ON THE HENSTOCK INTEGRAL

DONG IL RIM AND WON KYU KIM

ABSTRACT. In this paper we prove a controlled convergence theorem for the Henstock integral by using new conditions.

1. Introduction

In the 1950's J. Kurzweil and R. Henstock independently gave a Riemann Complete type integral, called the Kurzweil-Henstock integral (or KH-integral/H-integral). It has been proved that this integral is equivalent to the special Denjoy integral. Therefore the Henstock integral contains the Newton, Riemann and Lebesgue integrals. In 1985, P. Y. Lee and T. S. Chew [6, 7] gave the controlled convergence theorem. But we want to find a better convergence theorem. In this paper, using the $UACG_\delta$ property, we give a controlled convergence theorem.

First we introduce some necessary terms. Throughout this paper D will denote a finite collection of non-overlapping tagged intervals in $[a,b]$. For $D = \{(t_i, [c_i, d_i]) : 1 \le i \le N\}$, we will write

$$f(D) = \sum_{i=1}^{N} f(t_i)(d_i - c_i), \quad F(D) = \sum_{i=1}^{N}(F(d_i) - F(c_i)),$$

$$\text{and} \quad \mu(D) = \sum_{i=1}^{N}(d_i - c_i).$$

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Let δ be a positive function defined on $[a,b]$. We say that D is subordinate to δ if $[c_i,d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each i and that D is subordinate to δ on $[a,b]$ if in addition D is a partition of $[a,b]$. A real-valued function f is said to be Henstock integrable to A on a closed bounded interval $[a,b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that whenever a division D given by

$$a = x_0 < x_1 < \cdots < x_n = b$$

satisfies $0 \le x_i - \xi_i < \delta(\xi_i)$ and $0 \le \xi_i - x_{i-1} < \delta(\xi_i)$ for $i = 1, 2, \ldots, n$ we have

$$\Big| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \Big| < \varepsilon,$$

or alternatively,

$$\Big| \sum f(\xi)(v-u) - A \Big| < \varepsilon$$

where $[u,v]$ denotes a typical interval in D with $\xi - \delta(\xi) < u \le \xi \le v < \xi + \delta(\xi)$. Such a division D is said to be compatible with $\delta(\xi)$.

2. Preliminaries

DEFINITION 2.1. (a) A function on $[a,b]$ is ACG_δ^* on $X \subset [a,b]$ if X is the union of a sequence of subsets X_i such that F is $\mathrm{AC}_\delta^*(X_i)$ for each i, i.e., for every $\varepsilon > 0$ there are $\eta > 0$ and $\delta(\xi) > 0$ such that for any δ-fine partial division$D = \{([u,v],\xi)\}$ with $\xi \in X_i$ satisfying $(D)\sum |v-u| < \eta$ we have $(D)\sum |F(u,v)| < \varepsilon$.
(b) The sequence $\{F_n\}$ is UACG_δ^* on $X \subset [a,b]$ if X is the union of a sequence of subsets X_i such that $\{F_n\}$ is $\mathrm{UAC}_\delta^*(X_i)$ for each i, independent of n.

DEFINITION 2.2. Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$ and let $X \subset [a,b]$ be measurable.
(a) The sequence $\{F_n\}$ is $\mathcal{P}$-Cauchy on E if $\{F_n\}$ converges pointwise on X and if for each $\varepsilon > 0$ there exist a positive function δ on X and

a positive integer N such that $|F_n(D) - F_m(D)| < \varepsilon$ for all $m, n \geq N$ whenever D is X-subordinate to δ.
(b) The sequence $\{F_n\}$ is generalized $\mathcal{P}$-Cauchy on X if X can be written as a countable union of measurable sets on each of which $\{F_n\}$ is $\mathcal{P}$-Cauchy.

Define $f_X(x) = f(x)$ when $x \in X$ and $f_X(x) = 0$, otherwise.

THEOREM 2.1. *Let the following conditions be satisfied:*

(i) $f_{n,X}(X) \to f_X(x)$ *almost everywhere in* $[a, b]$ *as* $n \to \infty$ *where each* $f_{n,X}$ *is Henstock integrable on* $[a, b]$*;*
(ii) *the primitives* $F_{n,X}$ *of* $f_{n,X}$ *are* $UAC^*_\delta(X)$ *with closed set* X *in* $[a, b]$.

Then f_X *is Henstock integrable on* $[a, b]$ *with the primitive* F_X.

Proof. By (ii), for every $\varepsilon > 0$ there exist a $\delta(\xi) > 0$ and an $\eta > 0$, both independent of n, such that for any δ-fine partial division D of X satisfying

$$(D) \sum |v - u| < \eta \quad \text{we have} \quad (D) \sum |F_{n,X}(u, v)| < \varepsilon.$$

By Egoroff's theorem, there is an open set G with $|G| < \eta$ such that

$$|f_n(\xi) - f_m(\xi)| < \varepsilon \quad \text{for } n, m \geq N \text{ and } \xi \notin G.$$

Consider the following, in which D is a δ-fine division of $[x, y]$ and $D = D_1 \cup D_2$ so that D_1 contains the intervals with the associated points $\xi \notin G$ and D_2 otherwise,

$$\begin{aligned}
|F_{n,X}(x,y) - F_{m,X}(x,y)| &= \left|(D) \sum \{F_{n,X}(u,v) - F_{m,X}(u,v)\}\right| \\
&\leq (D_1) \sum |\, F_{n,X}(u,v) - f_{n,X}(\xi)(v-u)\,| \\
&\quad + (D_1) \sum |\, F_{m,X}(u,v) - f_{m,X}(\xi)(v-u)\,| \\
&\quad + (D_1) \sum |\, f_{n,X}(\xi) - f_{m,X}(\xi)\,|(v-u) \\
&\quad + (D_2) \sum |\, F_{n,X}(u,v)\,|
\end{aligned}$$

$$+(D_2)\sum |F_{m,X}(u,v)|$$
$$< \varepsilon(4+b-a).$$

Hence, for any partial division D of $[a,b]$ we have

$$\Big|(D)\sum\{F_{n,X}(u,v)-F_{m,X}(u,v)\}\Big|<\varepsilon \quad \text{for } n,m\geq N.$$

Therefore the sequence $\{F_{n,X}\}$ is generalized $\mathcal{P}$-Cauchy on $[a,b]$. By [2, Theorem 13.32], f_X is Henstock integrable on $[a,b]$ with the primitive F_X. □

DEFINITION 2.3. (a) A sequence $\{F_n\}$ of functions is uniformly-AC$^\nabla$ on X whenever to each $\varepsilon>0$ there exist $\eta>0$ and a gage $\delta : X \to \mathbb{R}^+$ such that

$$\sup_n\Big|\sum_{J_k\in P_1}F_n(J_k)-\sum_{L_h\in P_2}F_n(L_h)\Big|<\varepsilon, \tag{1}$$

for each $P_1, P_2 \in \prod(X;\delta)$ with

$$|(\cup P_1)\bigtriangleup(\cup P_2)|<\eta. \tag{2}$$

(b) A sequence $\{F_n\}$ of functions is uniformly-ACG$^\nabla$ on $[a,b]$ if $[a,b]=\cup_i X_i$ where X_i are measurable sets and $\{F_n\}$ is uniformly-AC$^\nabla$ on each X_i.

PROPOSITION 2.2. *If $\{F_n\}$ is uniformly-ACG$^\nabla$, then $\{F_n\}$ is uniformly-ACG$^*_\delta$.*

Proof. Let $[a,b]=\cup_i X_i$ be such that $\{F_n\}$ is uniformly-AC$^\nabla$ on each X_i. So to each $\varepsilon>0$ there exist a constant $\eta>0$ and a gage δ on each X_i such that (1) holds for each $P_1,P_2\in\prod(X_i;\delta)$ satisfying condition (2). Take $D=\{([c_k,d_k],x_k)\}_{k=1}^p$ with $\sum_k|d_k-c_k|<\eta$ and put $P_1=\{([c_k,d_k],x_k) : F_n(c_k,d_k)\geq 0\}$, $P_2=\{([c_k,d_k],x_k) : F_n(c_k,d_k)<0\}$. So $|(\cup P_1)\bigtriangleup(\cup P_2)|=|\cup D|=\sum_{k=1}^p|d_k-c_k|<\eta$

and (2) holds. Then by (1)

$$\sup_n \sum_{k=1}^{p} |F_n(c_k, d_k)| = \sup\Big| \sum_{(c_k,d_k)\in P_1} F_n(c_k, d_k) - \sum_{(c_k,d_k)\in P_2} F_n(c_k, d_k) \Big| < \varepsilon$$

Hence $\{F_n\}$ is uniformly-ACG$_\delta^*$. □

We get following theorem by Theorem 2.1 and Proposition 2.2.

THEOREM 2.3. *Let the following conditions be satisfied:*

(i) $f_{n,X}(X) \to f_X(x)$ *almost everywhere in* $[a,b]$ *as* $n \to \infty$ *where each* $f_{n,X}$ *is Henstock integrable on* $[a,b]$;

(ii) *the primitives* $F_{n,X}$ *of* $f_{n,X}$ *are* $UAC^{\nabla}(X)$ *with closed set* X *in* $[a,b]$.

Then f_X *is Henstock integrable on* $[a,b]$ *with the primitive* F_X.

3. Controlled convergence theorem

DEFINITION 3.1. Let $F : [a,b] \to \mathbb{R}$ and let $X \subset [a,b]$. A function F is AC$_\delta$ on X if for each $\varepsilon > 0$ there exist a positive number η and a positive function δ on X such that $|F(D)| < \varepsilon$ whenever D is subordinate to δ, all of the tags of D are in X, and $\mu(D) < \eta$. A function F is ACG$_\delta$ on $[a,b]$ if $[a,b]$ is the union of a sequence of set $\{X_i\}$ such that the function F is AC$_\delta(X_i)$ for each i.

DEFINITION 3.2. Let $F : [a,b] \to \mathbb{R}$ and let $X \subset [a,b]$. A function F is $AC^*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_k, b_k]\}$ with $a_k, b_k \in X$

$$\sum_k |b_k - a_k| < \eta \quad \text{implies} \quad \sum_k \omega(F; [a_k, b_k]) < \varepsilon$$

where ω denotes the oscillation of F over $[a_k, b_k]$. A function F is ACG* on $[a,b]$ if F is continuous and $[a,b]$ is the union of a sequence of sets $\{X_i\}$ such that the function F is AC$^*(X_i)$ for each i.

PROPOSITION 3.1. *A function F is ACG_δ on $[a,b]$ if and only if it is ACG^* on $[a,b]$.*

Proof. See [3, Theorem 5, Theorem 6]. □

THEOREM 3.2. *The following conditions be satisfied:*

(i) *$f_n(x) \to f(x)$ almost everywhere in $[a,b]$ as $n \to \infty$ where each f_n is Henstock integrable on $[a,b]$;*
(ii) *the primitives F_n of f_n are $UACG_\delta$;*
(iii) *the sequence F_n converges uniformly to a continuous function F on $[a,b]$.*

Then f is Henstock integrable on $[a,b]$ with the primitive F.

Proof. By condition (ii) there exists a sequence $\{X_i\}$, $[a,b] = \cup_{i=1}^{\infty} X_i$ such that $F_n \in \mathrm{UAC}_\delta$ in X_i a bounded, closed set with bounds a and b and put $X = X_i$. Since $F_n \to F$ everywhere on $[a,b]$ by (iii), we have $F \in \mathrm{AC}_\delta$ on X and hence $F \in \mathrm{ACG}_\delta$ on $[a,b]$. By Proposition 3.1, $F \in \mathrm{AC}^*$ on X and hence $F \in \mathrm{ACG}^*$ on $[a,b]$ and also $F \in \mathrm{AC}$ on X and hence $F \in \mathrm{ACG}$ on $[a,b]$. Now we prove $F'(x) = f(x)$ almost everywhere on $[a,b]$. Let $G_n : [a,b] \to \mathbb{R}$ equal F_n on X, and extend G_n linearly to the closed intervals contiguous to X. Likewise we define G from F. We see that G_n and G are uniformly absolutely continuous on $[a,b]$. First we will prove $G'(x) = f(x)$ almost everywhere on X. By (iii) $G_n(x) \to G(x)$ on $[a,b]$. Let $[c_k, d_k]$, $k = 1, 2, \cdots$ be the intervals contiguous to X. Then we have $|G_n'(x)| \le M_k$, $x \in (c_k, d_k)$, $n = 1, 2, \ldots$, and hence for $k = 1, 2, \ldots$, $\{G_n\} \in \mathrm{AC}([c_k, d_k])$ uniformly. Further,

$$G_n'(x) = \frac{G_n(d_k) - G_n(c_k)}{d_k - c_k}, \quad x \in (c_k, d_k).$$

Consequently, $G_n'(x)$ converges on (c_k, d_k), $k = 1, 2, \ldots$. Hence G_n' converges on $[a,b]$ almost everywhere. Since $\{G_n\} \in \mathrm{AC}$ on X uniformly, then $G_n'(x) = f_n(x)$ on X and $G_n'(x) = f_n(x) \to f(x)$ on X. Therefore by [7] Corollary 7.7, $G'(x) = g(x) = f(x) = F'(x)$ almost everywhere on X. Thus $F'(x) = f(x)$ almost everywhere on $[a, b]$ by Theorem 2.1. Therefore there exists an ACG_δ function F on $[a, b]$ such that $F' = f$ almost everywhere on $[a, b]$. Hence f is Henstock integrable on $[a, b]$ with the primitive F by [2, Theorem 9.17]. □

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DONG IL RIM
DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU 361-763, KOREA

WON KYU KIM
DEPARTMENT OF MATHEMATICS EDUCATION, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU 361-763, KOREA