JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **12**, August 1999

EXISTENCE OF SOLUTIONS FOR THE NONLINEAR HYPERBOLIC SYSTEM OF CONSERVATION LAWS IN SEVERAL SPACE VARIABLES

CHOON HO LEE*

ABSTRACT. In this paper we prove the existence of solutions for the nonlinear hyperbolic system of conservation laws in several space variables provided that the initial data are very close to each other. We also give an example and discuss the existence of solutions for large initial data

1. Introduction

In this paper we will study the existence of solutions for the nonlinear hyperbolic system of conservation laws in several space variables

$$u_t + \sum_{i=1}^{n} f^i(u)_{x_i} = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1}$$

where $u = (u_1, u_2, \cdots, u_N)$ is a vector function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, and $f^i = (f^i_1, f^i_2, \ldots, f^i_N)$ is a vector function of $u \in \mathbb{R}^N$. The Riemann problem is an initial value problem of (1) with a given initial data

$$u(x,0) = \begin{cases} u_r, & \nu \cdot x > 0, \\ u_l, & \nu \cdot x < 0, \end{cases} \tag{2}$$

*This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1997.

Received by the editors on June 30, 1999.

1991 *Mathematics Subject Classifications* : Primary 35L45, 35L65 Secondary 76L05.

Key words and phrases: Conservation Laws, Shock Planar Waves, Planar Centered Wave, Contact Discontinuity.

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is a given unit vector. In one space dimension case, the existence of solutions of equation (1) is obtained by several authors [1], [6], [9] and therein. Many years ago, several authors had studied in this area, Conway and Smoller[2], Kruzkov[5], and Vol'pert[10] proved the existence and uniqueness of weak solutions if $N = 1$ and initial data $u_0(x)$ is bounded and of bounded variation in the sense of Tonelli-Cesari. Recently, Majda[7], Dafermos[3],[4] and Rauch[8] also study the system of conservation laws in several space dimensions. Zhang and Hsiao[1] also classified the solutions of the scalar equation of conservation laws in two dimensional spaces when the initial data is differently given on each quadrant.

In this paper we will prove the existence of solutions which satisfies some additional conditions provided that initial data are very small and give an example in gas dynamics.

This paper consists of four sections: In the second section we shall give some terminologies which will be use the next sections. In the third section, we will look into the existence of planar centered waves, shock waves satisfying the jump conditions and the stability conditions, contact discontinuities occurring in the degenerate(or nonconvex) system, and prove our main theorem which is the existence of solutions in the Riemann problem. In the fourth section, we will give an example appearing the gas dynamics in the Lagrangian coordinate system of the two dimension case. Finally we introduce results and their methods of the proofs of Conway and Smoller[2], Vol'pert[10], and Kruzkov[5].

2. Preliminaries

In this section we give some terminologies which will use next sections. The function $u(x,t)$ is a *weak solution* of the system of conservation laws (1) with initial value $u_0(x)$ if u and $f(u) = (f^1(u), \cdots, f^n(u))$ are integrable functions over every bounded set of the half-plane $t \geq 0$

and the integral relation

$$(3) \qquad \int_0^\infty \int_{\mathbb{R}^n} \{\phi_t u + \sum_{i=1}^n \phi_{x_i} f^i(u)\}\, dx\, dt + \int_{\mathbb{R}^n} \phi(x,0) u_0(x)\, dx = 0$$

is satisfied for all smooth test vectors $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^N)$.

For piecewise smooth solutions, the condition (3) are equivalent to requiring the following two properties :

I. In regions of smoothness for u, the equation in (1) are satisfied,

II. If u has a jump discontinuity across a space-time hypersurface Γ with outward space-time normal (s, ν), $\nu = (\nu_1, \ldots, \nu_n)$, the following nonlinear boundary conditions are satisfied,

$$(4) \qquad s[u] + \sum_{i=1}^n \nu_i [f^i(u)]|_\Gamma = 0$$

with the brackets $[\bullet]$ denoting the jump in a quantity.

The conditions in (4) are called the *generalized Rankine-Hugoniot jump conditions.*

As we know that the Riemann problem (1) and (2) is invariant to the dilation $(x,t) \to (\alpha x, \alpha t)$, we will consider a solution of the form

$$(5) \qquad u = u(\xi), \quad \xi = \frac{\nu \cdot x}{t}$$

which is modified by the 1-dimensional case. If (5) is a solution of (1) and (2), (5) is a solution of the system of ordinary differential equations

$$-\xi du + d(\nu \cdot f(u)) = 0$$

with the boundary condition

$$u(+\infty) = u_r, u(-\infty) = u_l,$$

where $\nu \cdot f(u) = \sum_{i=1}^n \nu_i f^i(u)$ and $f(u) = (f^1(u), \ldots, f^n(u))$ is a matrix. This implies the nonlinear eigenvalue value problem

$$(6) \qquad (\nu \cdot Df(u) - \xi I) du = 0,$$

where $\nu \cdot Df(u) = \sum_{i=1}^{n} \nu_i Df^i(u)$ and Df^i is a Jacobi matrix of f^i. This idea gives us the following definition: A system (1) is called *hyperbolic in the direction* ν if the matrix $\nu \cdot Df(u)$ has n real eigenvalues $\lambda_i(u;\nu)(i = 1, 2, \cdots, n)$. It is called *strictly hyperbolic in the direction* ν if all of the $\lambda_i(u;\nu)$ are distinct, i. e.

$$\lambda_1(u;\nu) < \lambda_2(u;\nu) < \cdots < \lambda_n(u;\nu).$$

The system (1) is called *(strictly) hyperbolic* if it is strictly hyperbolic in every direction ν.

We assume (1) is strictly hyperbolic in the direction ν. Then the singular solution of (6) will be expressed by

$$\lambda = \lambda_i(u;\nu) \tag{7}$$

$$du \text{ is parallel to } r_i(u;\nu) \tag{8}$$

where $r_i(u;\nu)$ is the right eigenvector corresponding to the eigenvalue $\lambda_i(u;\nu)$.

An *i-Riemann invariant in the direction* ν is a smooth function $w_\nu : N \to \mathbb{R}$ such that if $u \in N$,

$$r_i(u;\nu) \cdot \nabla_u w_\nu(u) = 0. \tag{9}$$

Then there are $(n-1)$ i-Riemann invariants whose gradients are linearly independent in N.

A system (1) is called *i-th convex or genuinely nonlinear in the direction* ν if

$$r_i(u;\nu) \cdot \nabla_u \lambda_i(u;\nu) \neq 0 \tag{10}$$

and is called *i-th convex or genuinely nonlinear* if it is i-th convex or genuinely nonlinear in every direction ν. If this is the case, we shall normalize $r_i(u;\nu)$ by $r_i(u;\nu) \cdot \nabla_u \lambda_i(u;\nu) = 1$.

The system is called *convex or genuinely nonlinear* if it is i-th convex or genuinely nonlinear for each $i(1 \leq i \leq n)$.

3. The Proof of Main Theorem

Let u be a C^1 solution of (1) in a domain N, and suppose that all i-Riemann invariants in the direction ν are constant in N. We call u an *i-rarefaction wave in the direction ν*(or an *i-simple wave in the direction ν*). A *planar centered wave with central plane* at centered at (x_0, t_0) is a simple wave depending only on $\frac{\nu\cdot(x-x_0)}{t-t_0}$. Suppose that the i-th characteristic field is genuinely nonlinear in N in the direction ν, and let $u_l \in N$. There exists a smooth one-parameter family of states $u(\xi)$, defined for $|\xi|$ sufficiently small, which can be connected to u_l on the right by a i-planar centered wave. Since the i-Riemann invariants $w_{\nu i}$ are constant, we have $w_{\nu i}(u) = w_{\nu i}(u_l)$, $i = 1, 2, \ldots, n-1$. We introduce a parameter τ by $\lambda_i(u;\nu) = \lambda_i(u_l;\nu) + \tau$. Define a function $F_\nu : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$\begin{aligned} F_\nu(u,\tau) = \quad & (w_{\nu 1}(u) - w_{\nu 1}(u_l), \ldots, w_{\nu n-1}(u) - w_{\nu n-1}(u_l), \\ & \lambda_i(u;\nu) - \lambda_i(u_l;\nu) - \tau), \quad u \in \mathbb{R}^n. \end{aligned}$$

By the implicit function theorem, the equation $F_\nu(u,\tau) = 0$ defines a curve $u = u(\tau; u_l, \nu)$ depending on u_l, for $|\tau|$ sufficiently small.

THEOREM 3.1. *Let the i-th characteristic field of the system (1) be genuinely nonlinear in N in the direction ν, and normalized so that $\nabla\lambda_i \cdot r_i = 1$. Let u_l be any point in N. There exists an one-parameter family of states $u = u(\tau)$, $0 \le \tau < \epsilon$, $u(0) = u_l$, which can be connected to u_l on the right by an ith planar centered wave. Moreover the parameterization can be chosen so that $\frac{du}{d\tau} = r_i$ and $\frac{d^2u}{d\tau^2} = \frac{dr_i}{d\tau}$.*

It is obvious that a state u_r can be joined to u_l by an i-th planar centered wave with central line $\nu \cdot (x - x_0) = 0, t = t_0$ if and only if u_r is located on $R_i(u_l;\nu)$, the integral curve of (7) with initial data u_l in the u-space, and $\lambda_i(u;\nu)$ increases when u varies from u_l to u_r along $R_i(u_l;\nu)$. Changing direction ν, $R_i(u_l;\nu)$ will form a horn-cone with vertex u_l, called the *rarefaction wave cone* denoted by $R_i(u_l)$. In the case when (1) is i-th convex, the horn-cone $R_i(u_l)$ is divided into two parts by the vertex u_l, and u_r can be joined to u_l by an i-th planar centered wave if and only if u_r belongs to a part of $R_i(u_l)$ on which $\lambda_i(u_r;\nu) > \lambda_i(u_l;\nu)$.

On the discontinuity plane

$$s = \frac{\nu \cdot x}{t} \tag{11}$$

the generalized Rankine-Hugoniot jump conditions (4) implies

$$s[u] = \nu \cdot [f] \tag{12}$$

where $[u] = u_r - u_l$ and $[f] = f(u_r) - f(u_l)$. The normal vector $\vec{n} = (-s,\nu)$ of (11) is chosen to point toward the side of u_r.

For any fixed u_l, ν, we consider the system

$$s(u - u_l) = \nu \cdot (f(u) - f(u_l)) \tag{13}$$

which contains n equations but $n+1$ unknown variables $u = (u_1, \cdots, u_n)$ and s.

We assume that if the system (1) is genuinely nonlinear in the i-th characteristic field, so that $\nabla\lambda_i \cdot r_i \neq 0$, then r_i is normalized by $\nabla\lambda_i \cdot r_i = 1$ and l_i is normalized by $l_i r_i = 1$.

THEOREM 3.2. *Let the system (1) be hyperbolic in N, and let $u_l \in N$. Then there are n smooth one-parameter family of states $u = u_i(\tau)$, $i = 1, 2, \ldots, n$ defined for $|\tau| < \epsilon_i$, where $u_i(0) = u_l$, all of which satisfy the jump condition (12).*

Thus the solution should be an 1-dimensional manifold expressed as

$$\begin{aligned} u &= u(\tau; u_l, \nu) \\ s &= s(\tau; u_l, \nu) \end{aligned} \tag{14}$$

where the parameter τ is chosen so that

$$u(0; u_l, \nu) = u_l.$$

We denote the curve $u = u(\tau; u_l, \nu)$ in the u-space as $S(u_l; \nu)$. Theorem 3.2 shows that $S(u_l; \nu)$ consists of n branches in a neighborhood of u_l, $S_i(u_l; \nu)$ called the *i-shock curve*, $(i = 1, 2, \dots, n)$.

COROLLARY 3.2.1. *Along the i-th shock curve, if the i-th characteristic field is genuinely nonlinear, we can choose a parameterization so that $\frac{du_i}{d\tau}(0) = r_i$, and $\frac{d^2u}{d\tau^2}(0) = \frac{dr_i}{d\tau}$, where $r_i = r_i(u_l)$. Moreover, with this parameterization $s(0) = \lambda_i(u_l)$, and $\frac{ds}{d\tau}(0) = \frac{1}{2}$.*

THEOREM 3.3. *(a) The shock speed of an i-shock is the aritheoremetic average of the i-characteristic speeds on both sides of the shock, up to second-order terms in τ.*

(b) The change in an i-Riemann invariant across an i-shock is of third order in τ.

Theorem 3.3 shows that each branch $S_i(u_l; \nu)$ is tangential to $R_i(u_l; \nu)$ $(i = 1, 2, \dots, n)$ up to the second order if the system (1) is strictly hyperbolic in the direction ν. Changing direction ν, $S_i(u_l; \nu)$ will form a horn-cone with vertex u_l, called the *shock wave cone*, denoted by $S_i(u_l)$. Any u_r which can be joined to u_l by the i-th discontinuity plane must be on $S_i(u_l)$ $(1 \leq i \leq n)$.

When (1) is i-th convex, in addition to $u_r \in S_i(u_l)$, the discontinuity plane must satisfy the stability conditions:

$$\lambda_i(u_r; \nu) < s_i(\tau_r; u_l, \nu) < \lambda_i(u_l; \nu) \tag{15}$$

$$\lambda_{i-1}(u_l; \nu) \leq s_i(\tau_r; u_l, \nu) \leq \lambda_{i+1}(u_r; \nu), \tag{16}$$

where τ_r is defined so that $u(\tau_r; u_l, \nu) = u_r$, $\tau_r > 0$.

A discontinuity will be called a *shock wave* if it satisfies the Rankine-Hugoniot condition and the stability conditions (15), (16) if (1) is convex.

THEOREM 3.4. *The stability conditions (15) and (16) hold along the curve $u = u_k(\tau)$ if and only if $\tau < 0$.*

Proof. From Theorem 3.2 we can write $\lambda_j(\tau) = \lambda_j(u_k(\tau);\nu)$ and $s_j(\tau;\nu) = s_j(u_k(\tau);\nu)$. The stability conditions (9) and (10) can be written as

(i) $\lambda_i(\tau;\nu) < s_i(\tau;\nu) < \lambda_i(0;\nu)$,

(ii) $\lambda_{i-1}(0;\nu) \leq s_i(\tau;\nu) \leq \lambda_{i+1}(\tau;\nu)$.

Let $\phi(\tau;\nu) = \lambda_i(\tau;\nu) - s_i(\tau;\nu)$. Then $\phi(0;\nu) = 0$, $\phi'(0;\nu) = \nabla\lambda_i \cdot r_k - \frac{ds_i}{d\tau}(0) = 1 - \frac{1}{2} > 0$. Thus if (1) holds we see $\tau < 0$. On the other hand, if $\tau < 0$, then we see $\phi(\tau;\nu) < 0$ so $\lambda_i(\tau;\nu) < s_i(\tau;\nu)$. Also $\frac{ds_i}{d\tau}(0) = \frac{1}{2}$ and $\lambda_i(0;\nu) = s_i(0;\nu)$ imply $\lambda_i(0;\nu) > s_i(\tau;\nu)$. Since $s_i(\tau;\nu) \to \lambda_i(0;\nu) > \lambda_{i-1}(0;\nu)$ as $\tau \to 0$, we have $s_i(\tau;\nu) > \lambda_{i-1}(0;\nu)$ for small τ. Finally $\lambda_{i+1}(0;\nu) > \lambda_i(0;\nu) = s_i(0;\nu)$ gives $\lambda_{i+1}(\tau;\nu) > s_i(\tau;\nu)$ for small τ. This completes the proof. □

Define a *composite curves* through $u_l \in N$ with the direction ν as follows: For each i, $1 \leq i \leq n$

$$U_i(\tau;\nu) = \begin{cases} \bar{u}_i(\tau;\nu), & \tau \leq 0, \\ \tilde{u}_i(\tau;\nu), & \tau \geq 0, \end{cases} \tag{17}$$

where $\bar{u}_i$ is the i-shock curve, and $\tilde{u}_i$ is the i-planar wave curve. Then Theorem 3.1 and Corollary 3.2.1 yield the following theorem.

THEOREM 3.5. *The curves $U_i(\tau;\nu)$, $k = 1, 2, \dots, n$, have two continuous derivatives at τ.*

Changing the direction ν, we have two cones which are composed of a shock wave cone and a rarefaction wave cone. In a general case, a discontinuity surface in the space (x,t) of a solution of (1) can be approximated in a neighborhood of any regular point by a tangential plane at that point.

A system (1) is called *i-th linear degenerate(or nonconvex) in the direction ν* if

$$r_i(u;\nu) \cdot \nabla_u \lambda_i(u;\nu) = 0 \tag{18}$$

Then by definition, $\lambda_i(u;\nu)$ is an i-Riemann invariant. Thus if $u(\tau)$, $|\tau| < \epsilon$, is the solution of the problem

$$\frac{du}{d\tau} = r_k(u(\tau);\nu), \quad u(0) = u_l,$$

then $\frac{d\lambda_i}{d\tau} = \nabla\lambda_i \cdot r_k = 0$ implies that λ_i is constant along this curve; i.e., $\lambda_i(u(\tau);\nu) = \lambda_i(u(0);\nu) = \lambda_i(u_l;\nu)$, $|\tau| < \epsilon$. Now if $|\tau| < \epsilon$, define a function $v(x,t;\nu)$ by

$$v(x,t;\nu) = \begin{cases} u_l, & \xi < \lambda_i(u_l), \\ u(\tau;\nu), & \xi > \lambda_i(u), \end{cases}$$

where $\xi = \frac{x\cdot\nu}{t}$. Then v is a solution of (1) with the initial condition

$$u(x,0) = \begin{cases} u_l, & \nu\cdot x < 0, \\ u(\tau;\nu), & \nu\cdot x > 0. \end{cases} \tag{19}$$

LEMMA 3.6. *Let v is a solution of (1) and (19). Then v satisfies the following jump condition:*

$$\nu\cdot[f] = s[u], \tag{20}$$

where $[f] = f(u(\tau;\nu)) - f(u_l)$, $[u] = u(\tau;\nu) - u_l$, and $s = \lambda(u_l;\nu)$, on the hyperplane $\xi = \lambda_i(u_l;\nu)$.

A solution u of (1) is called a *contact discontinuity* in the direction ν if the shock speed equals the characteristic speed on one side. The Lemma 3.6 gives

THEOREM 3.7. *If two nearby states u_l and u_r have the same i-Riemann invariants with respect to a linearly degenerate field, then they are connected to each other by a contact discontinuity of speed $s = \lambda_i(u_l;\nu) = \lambda_i(u_r;\nu)$.*

When (1) is not i-th $(i = 1, 2, \ldots, n)$ convex the discontinuity plane has to satisfy the following stability conditions:

$$s_i(\tau; u_l, \nu) \geq s_i(\tau_r; u_l, \nu), \quad \tau \in [0, \tau_r] \tag{21}$$

$$\lambda_{i-1}(u_l; \nu) \leq s_i(\tau_r; u_l, \nu) \leq \lambda_{i+1}(u_r; \nu) \tag{22}$$

Now we will prove the initial value problem which is called the Riemann problem. In order to prove this, we assume that (1) is strictly hyperbolic, and that in N each characteristic field is either genuinely nonlinear or linearly degenerate. Moreover we also assume that $|\tau|$ is so small that the curves $U_i(\tau; \nu)$ defined by (17) all exist provided that the ith characteristic field is genuinely nonlinear, and that if the ith characteristic field is linearly degenerate, the curves satisfying $dv/d\tau = r_k(u_k(\tau))$ all exist.

THEOREM 3.8. *Let $u_l \in N$ and suppose that the system (1) is hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate in N. Then there is a neighborhood $N_0 \subset N$ of u_l such that if $u_r \in N_0$, then Riemann problem (1) and (2) has a solution. This solution consists of at $(n+1)$-constant states separated by shocks, rarefaction waves, or contact discontinuities.*

Proof. From Theorem 3.5 and Theorem 3.7, for each $i = 1, 2, \ldots, n$, there exists an one-parameter family of transformations

$$L^i_{\tau_i} : N \to \mathbb{R}^n, \qquad |\tau_i| < \epsilon,$$

which is C^2 in τ_i, with the property that any $u \in N$ can be joined to $L^i_{\tau_i} u$ on the right by either a shock, rarefaction wave, or contact discontinuity depending on the i-characteristic field and the sign of $|\tau|$.

Let u_l be any point in N, and define $U = \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}^n : |\tau_i| < \epsilon, i = 1, \ldots, n\}$. We consider the composition transformation $L : U \to \mathbb{R}^n$ given by $L(\tau) = L^n_{\tau_n} L^{n-1}_{\tau_{n-1}} \cdots L^1_{\tau_1} u_l$, $\tau = (\tau_1, \ldots, \tau_n)$. Our goal is to show that there is $\bar{\tau}$ in U such that $T(\bar{\tau}) u_l = u_r$, provided that $|u_r - u_l|$ is small. To this end, we define a mapping $F : U \to \mathbb{R}^n$ by

$$F(\tau_1, \dots, \tau_n) = L^n_{\tau_n} L^{n-1}_{\tau_{n-1}} \cdots L^1_{\tau_1} u_l - u_l.$$

Then $F(0, \dots, 0) = 0$, and since

$$L^i_{\tau_i} u = u + \tau_i r_i(u; \nu) + O(\tau_i^2), k = 1, \dots, n,$$

we have

$$F(\tau_1, \dots, \tau_n) = \sum_{i=1}^{n} \tau_i r_i(u_l; \nu) + O(\tau_i^2).$$

This implies that $DF(0, \dots, 0) = (r_1(u; \nu), \dots, r_n(u; \nu))$. Since this latter matrix is nonsingular, the inverse function theorem shows that F is a homeomorphism of a neighborhood of $\tau = 0$ onto a neighborhood of $u = 0$. Therefore, if $|u_r - u_l|$ is small, there is a unique $\bar{\tau} = (\bar{\tau_1}, \dots, \bar{\tau_n})$ such that $F(\bar{\tau_1}, \dots, \bar{\tau_n}) = u_r - u_l$. That is,

$$L^n_{\tau_n} L^{n-1}_{\tau_{n-1}} \cdots L^1_{\tau_1} u_l - u_l = u_r - u_l,$$

or

$$L^n_{\tau_n} L^{n-1}_{\tau_{n-1}} \cdots L^1_{\tau_1} u_l = u_r.$$

This completes the proof. □

4. Example

In this section we will consider the example of isothermal flow in gas dynamics. The flow is modeled by the following system of conservation laws:

$$\left\{ \begin{array}{l} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0 \end{array} \right. \tag{23}$$

where (u, v), $p > 0$ and $\rho > 0$ represent velocity, pressure and density respectively, $p = p(\rho)$ is a given state function which is of the form $p = \rho^\gamma$ in the case of a polytropic gas, $\gamma > 1$ is the so-called adiabatic exponent constant. For a smooth solution, this equation (14) can be rewritten as

(24)

$$\begin{pmatrix} u \\ v \\ \rho \end{pmatrix}_t + \begin{pmatrix} u & 0 & p'(\rho)/\rho \\ 0 & u & 0 \\ \rho & 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho \end{pmatrix}_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & p'(\rho)/\rho \\ 0 & \rho & v \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho \end{pmatrix}_y = 0$$

From (6) and (24), the characteristic equation

$$\begin{vmatrix} \mu u + \nu v - \lambda & 0 & \mu p'(\rho)/\rho \\ 0 & \mu u + \nu v - \lambda & \nu p'(\rho)/\rho \\ \mu\rho & \nu\rho & \mu u + \nu v - \lambda \end{vmatrix} = 0$$

gives the three characteristics

$$\lambda_0 = \mu u + \nu v,$$

$$\lambda_\pm = \mu u + \nu v \pm \sqrt{p'(\rho)}.$$

Thus (23) is a strictly hyperbolic in the plane (u, v, ρ) with $\rho > 0$. The characteristics λ_0 and $\lambda_\pm$ are called *flow characteristic* and *wave characteristics*, respectively. The eigenvector r_0 corresponding to flow characteristics is

$$r_0 = (-\nu, \mu, 0).$$

Since $r_0 \cdot \nabla\lambda_0 = 0$, (23) is linear degenerate corresponding to flow characteristics. The rarefaction wave $R_0(u_l, v_l, \rho_l; \mathbf{n})$ in the direction $\mathbf{n} = (\mu, \nu)$ is a straight line:

$$\begin{cases} \mu(u - u_l) + \nu(v - v_l) = 0 \\ \rho - \rho_l = 0. \end{cases}$$

Changing the vector $\mathbf{n}$, we have the rarefaction wave cone $R_0(u_l, v_l, \rho_l)$ which is the plane

(25) $$\rho = \rho_l.$$

For eigenvector $r_\pm = (\pm\mu, \pm\nu, \frac{\rho}{\sqrt{p'(\rho)}})$ corresponding to $\lambda_\pm$, the relation

$$r_\pm \cdot \nabla\lambda_\pm = \pm\frac{\rho p''(\rho) + 2p'(\rho)}{2p'(\rho)} \tag{26}$$

implies that (27) is convex corresponding to $\lambda_\pm$ if $\rho p'' + 2p' > 0 \quad (\rho > 0)$. The rarefaction wave cone $R_\pm(u_l, v_l, \rho_l; \mathbf{n})$ can be expressed as

$$\pm\mu\int_{\rho_l}^{\rho}\frac{\sqrt{p'(\tau)}}{\tau}\,d\tau = u - u_l, \quad \pm\nu\int_{\rho_l}^{\rho}\frac{\sqrt{p'(\tau)}}{\tau}\,d\tau = v - v_l. \tag{27}$$

Changing the vector $\mathbf{n}$, we have the rarefaction wave cone $R_\pm(u_l, v_l, \rho_l)$

$$(u - u_l)^2 + (v - v_l)^2 = \left(\int_{\rho_l}^{\rho}\frac{\sqrt{p'(\tau)}}{\tau}\,d\tau\right)^2. \tag{28}$$

Rankine-Hugoniot jump condition

$$\begin{cases} s(\rho - \rho_l) & = \mu(\rho u - \rho_l u_l) + \nu(\rho v - \rho_l v_l) \\ s(\rho u - \rho_l u_l) & = \mu(\rho u^2 - \rho u_l^2 + p(\rho) - r(\rho_l)) + \nu(\rho uv - \rho_l u_l v_l) \\ s(\rho v - \rho_l v_l) & = \mu(\rho uv - \rho_l u_l v_l) + \nu(\rho v^2 - \rho v_l^2 + p(\rho) - r(\rho_l)) \end{cases}$$

gives the relation

$$\begin{pmatrix} \mu\rho & \nu\rho & \mu u_l + \nu v_l - s \\ \rho_l(\mu u_l + \nu v_l - s) & 0 & \mu[p]/[\rho] \\ 0 & \rho_l(\mu u_l + \nu v_l - s) & \nu[p]/[\rho] \end{pmatrix}\begin{pmatrix} u - u_l \\ v - v_l \\ \rho - \rho_l \end{pmatrix} = 0.$$

The shock wave $S_0(u_l, v_l, \rho_l; \mathbf{n})$ and $S_\pm(u_l, v_l, \rho_l; \mathbf{n})$ is

$$\begin{cases} \mu(u - u_l) + \nu(v - v_l) = 0 \\ \rho - \rho_l = 0 \\ s = s_0 = \mu u_l + \nu v_l \end{cases} \tag{29}$$

and

$$(30)\qquad \begin{cases} u-u_l=\pm\mu(\rho-\rho_l)\left(\frac{1}{\rho\rho_l}\times\frac{p-p_l}{\rho-\rho_l}\right)^{1/2}, \\ v-v_l=\pm\nu(\rho-\rho_l)\left(\frac{1}{\rho\rho_l}\times\frac{p-p_l}{\rho-\rho_l}\right)^{1/2}, \\ s_\pm=\pm\mu u_l+\nu v_l\pm\left(\frac{\rho}{\rho_l}\times\frac{p-p_l}{\rho-\rho_l}\right)^{1/2}, \end{cases}$$

respectively. We note that $S_0(u_l,v_l,\rho_l;\mathbf{n})$ (resp. $S_0(u_l,v_l,\rho_l)$) is the same as $R_0(u_l,v_l,\rho_l;\mathbf{n})$ (resp. $R_0(u_l,v_l,\rho_l)$). The shock wave cone $S_\pm(u_l,v_l,\rho_l)$ are the same circular cone

$$(31)\qquad (u-u_l)^2+(v-v_l)^2=\frac{1}{\rho\rho_l}(p-p_l)(\rho-\rho_l).$$

If the Rankine-Hugoniot condition hold and $(u_r,v_r,\rho_r)\in S_\pm(u_l,v_l,\rho_l)$, then the stability condition

$$(32)\qquad \lambda_\pm(u_r,v_r,\rho_r;\mathbf{n})<s_\pm<\lambda_\pm(u_l,v_l,\rho_l;\mathbf{n})\quad(\rho_r\lesseqgtr\rho_l)$$

holds. For convenience, we will consider only the case $p(\rho)=\rho^2$. Then from the relation (24), (27), (29) and (30), shock wave and rarefaction wave in the direction $\mathbf{n}=(\mu,\nu)$ are of the form

(33)

$$S_0(u_l,v_l,\rho_l;\mathbf{n})=R_0(u_l,v_l,\rho_l;\mathbf{n})=\{(u,v,\rho)\ :\ \mu(u-u_l)+\nu(v-v_l)=0,\rho-\rho_l=0\}$$

$$(34)\quad R_\pm(u_l,v_l,\rho_l;\mathbf{n})=\{(u,v,\rho)\ :\ \pm\mu\sqrt{8}(\rho^{1/2}-\rho_l^{1/2})=u-u_l,\ \pm\nu\sqrt{8}(\rho^{1/2}-\rho_l^{1/2})=v-v_l\}$$

and

$$S_\pm(u_l, v_l, \rho_l; \mathbf{n}) = \{\ (u, v, \rho)\ : u - u_l = \pm\mu(\rho - \rho_l)\sqrt{\frac{\rho + \rho_l}{\rho\rho_l}}, \tag{35}$$

$$v - v_l = \pm\nu(\rho - \rho_l)\sqrt{\frac{\rho + \rho_l}{\rho\rho_l}}\ \} \tag{36}$$

where $s_0 = \mu u_l + \nu v$ and $s_\pm = \mu u_l + \nu v_l \pm \left(\frac{\rho}{\rho_l}(\rho + \rho_l)\right)^{1/2}$. Changing the vector $\mathbf{n}$, we have shock wave cone and rarefaction wave cone:

$$S_0(u_l, v_l, \rho_l) = R_0(u_l, v_l, \rho_l) = \{(u, v, \rho)\ ;\ \rho = \rho_l\},$$

$$R_\pm(u_l, v_l, \rho_l) = \left\{(u, v, \rho)\ :\ (u - u_l)^2 + (v - v_l)^2 = 8(\rho^{1/2} - \rho_l^{1/2})^2\right\}$$

$$S_\pm(u_l, v_l, \rho_l) = \left\{(u, v, \rho)\ :\ (u - u_l)^2 + (v - v_l)^2 = \frac{(\rho^2 - \rho_l^2)(\rho - \rho_l)}{\rho\rho_l}\right\}.$$

Since $8(\rho^{1/2} - \rho_l^{1/2})^2 \le \dfrac{(\rho^2 - \rho_l^2)(\rho - \rho_l)}{\rho\rho_l}$, the rarefaction wave cone $R_\pm(u_l, v_l, \rho_l)$ is contained in the shock wave cone $S_\pm(u_l, v_l, \rho_l)$(Figure 3.1)

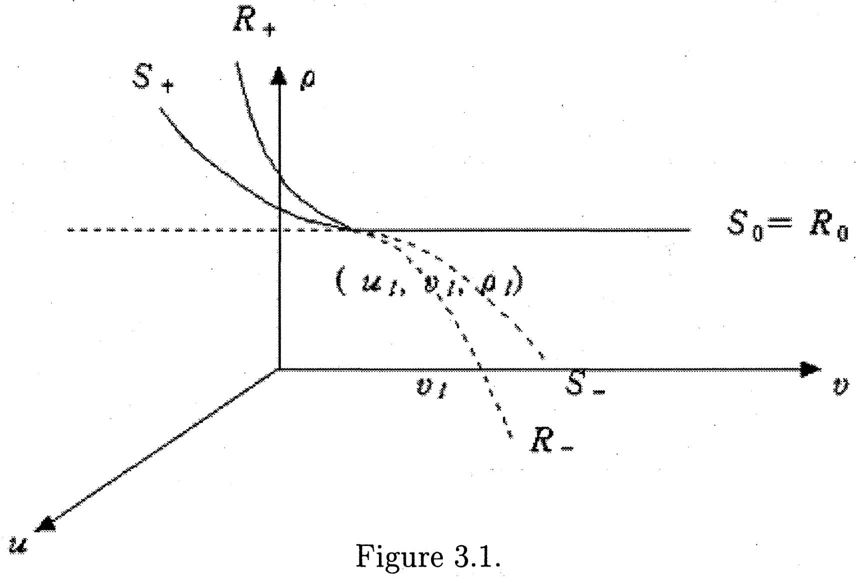

Figure 3.1.

Now we resolve the solution of the Riemann problem to (23) with initial data

(37)
$$(\rho, u, v)(x, y, 0) = \begin{cases} (\rho_l, u_l, v_l), & x > 0, \\ (\rho_r, u_r, v_r), & x < 0. \end{cases}$$

In this case, $\mathbf{n} = (1, 0)$. From (33), (34) and (35), we have

(38)
$$S_0(u_l, v_l, \rho_l; \mathbf{n}) = R_0(u_l, v_l, \rho_l; \mathbf{n}) = \{(u, v, \rho) \ : \ u = u_l, \rho - \rho_l = 0\}$$

$$R_\pm(u_l, v_l, \rho_l; \mathbf{n}) = \left\{(u, v, \rho) \ : \ \pm\sqrt{8}(\rho^{1/2} - \rho_l^{1/2}) = u - u_l, v = v_l\right\}$$

and

(39)
$$S_\pm(u_l, v_l, \rho_l; \mathbf{n}) = \left\{(u, v, \rho) \ : \ u - u_l = \pm(\rho - \rho_l)\sqrt{\frac{\rho + \rho_l}{\rho \rho_l}}, v = v_l\right\}$$

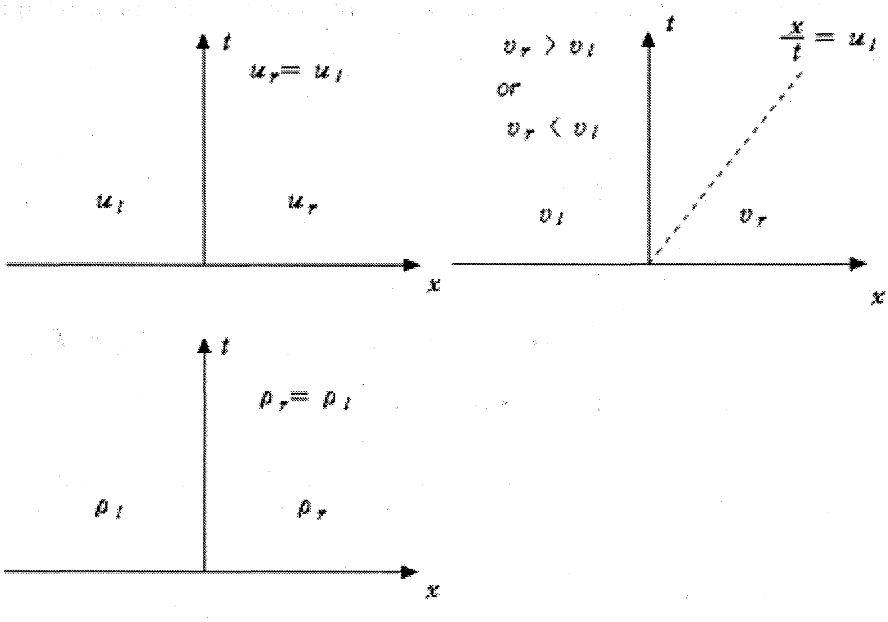

Figure 3.2.

where $s_0 = u_l$ and $s_\pm = u_l \pm (\frac{\rho}{\rho_l}(\rho + \rho_l))^{1/2}$. For convenience we will only consider three cases: (a) $(u_r, v_r, \rho_r) \in S_0 = R_0$, (b) $(u_r, v_r, \rho_r) \in S_+$, and (c) $(u_r, v_r, \rho_r) \in R_+$.

Case (a): If $(u_r, v_r, \rho_r) \in S_0 = R_0$, then $u_r = u_l$, $\rho_r = \rho_l$ and $v_r \leq v_l$(or $v_r \geq v_l$). From (29), we have $s_0 = u_l = \frac{x}{t}$. The solutions u and ρ are constant states for all times. But the solution v has a contact discontinuity with a speed $s_0 = u_l$. Thus solutions are depicted as Figure 3.2:

Case (b): If $(u_r, v_r, \rho_r) \in S_+$, then $u_r > u_l$, $\rho_r > \rho_l$ and $v_r = v_l$. From (30), we have $s_+ = u_l + \left(\frac{\rho}{\rho_l}(\rho + \rho_l)\right)^{1/2}$. The solution v is constant states for all times. But the solutions u and ρ have planar shock wave with a speed $s_+ = u_l + \left(\frac{\rho_r}{\rho_l}(\rho_r + \rho_l)\right)^{1/2}$. Thus solutions are depicted as Figure 3.3:

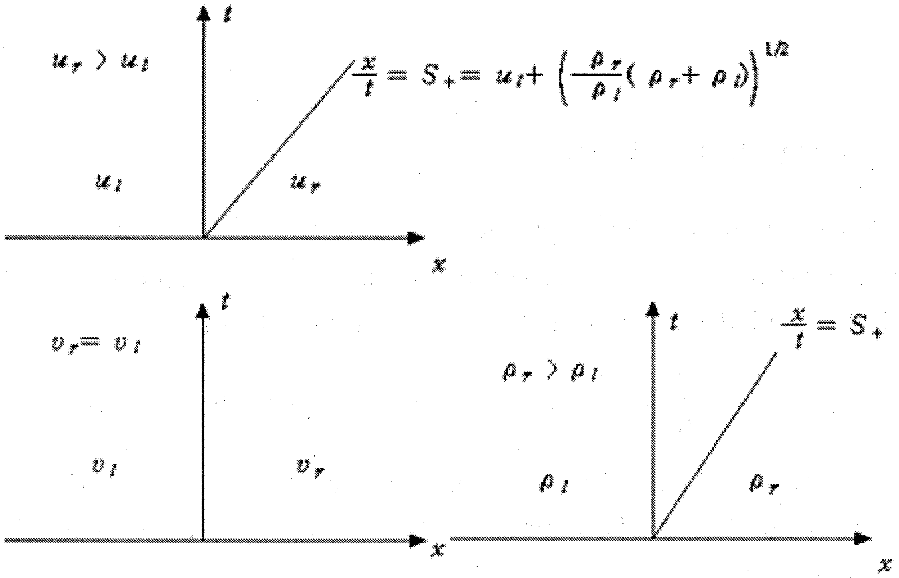

Figure 3.3.

Case (c): If $(u_r, v_r, \rho_r) \in R_+$, then $u_r > u_l$, $\rho_r > \rho_l$ and $v_r = v_l$. From Theorem 3.1, we have $\lambda_+(u_l) = u_l + \sqrt{2\rho_l} \leq \frac{x}{t} \leq u_r + \sqrt{2\rho_r} = \lambda_+$.

The solution v is constant states for all times. But the solutions u and ρ have planar rarefaction waves with a speed between $\lambda_+(u_l)$ and $\lambda_-(u_r)$. Thus solutions are depicted as Figure 3.4:

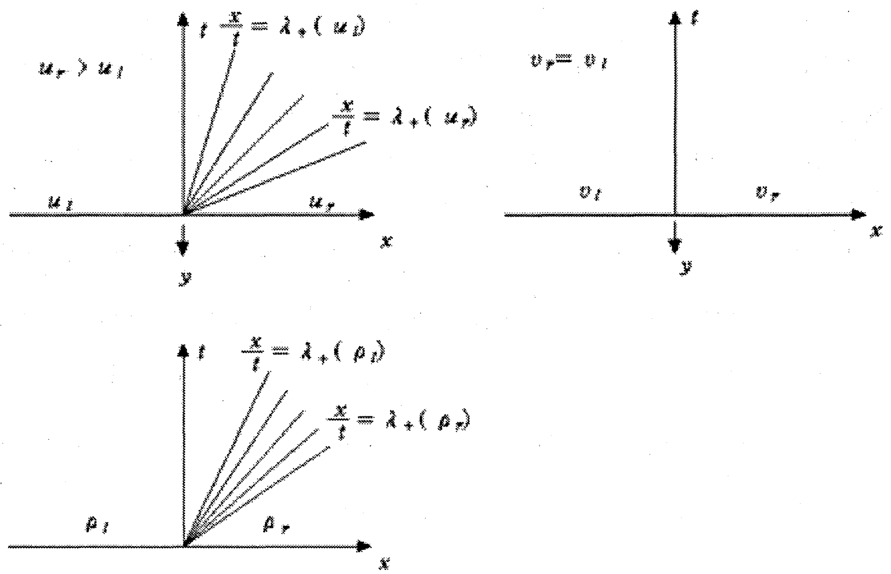

Figure 3.4.

4. Large Initial Data

Conway and Smoller[2] proved the existence of solutions of (1) in scalar case(i.e. $N = 1$) with the initial data $u(x, 0)$ which is bounded and of bounded variation in the sense of Tonelli-Cesari:

$$\int_{\Omega} |u(x+\mathbf{h}) - u(x)|\, dx \leq \text{ const} \cdot |\mathbf{h}| \tag{40}$$

for any compact Ω and any vector $\mathbf{h} \in \mathbb{R}^n$, where the constant does not depend on $\mathbf{h}$. Let the half-space $t \geq 0$ be covered by a grid defined by the planes

$$t = k\Delta t, x_i = m_i \Delta x_i, \quad i = 1, \cdots, n$$

where Δt and Δ_i are fixed positive real numbers, k runs through the non-negative integers, and m_i, $i = 1, \cdots, n$ assume all integral values.

In the region $t > 0$, they consider the finite difference method proposed by Lax and Oleinik[6]:

(41)

$$\frac{u_{\mathbf{m}}^{k+1} - \frac{1}{2n}\sum_{i=1}^{n}(u_{\mathbf{m}_i+1}^{k} + u_{\mathbf{m}_i-1}^{k})}{\Delta t} - \sum_{j=1}^{n} \frac{f^j(u_{\mathbf{m}_j+1}^{k}) - f^j(u_{\mathbf{m}_j-1}^{k})}{2\Delta x_j} = 0$$

where $\mathbf{m} = (m_1, \cdots, m_n)$, $\mathbf{m}_j \pm 1 = (m_1, \cdots, m_j \pm 1, \cdots, m_n)$ and $u_{\mathbf{m}}^k = u(m_1\Delta x_1, \cdots, m_n\Delta x_n, k\Delta t)$. They proved

THEOREM 4.1. [2] *Let f^i, $i = 1, \cdots, n$ be continuously differentiable functions of a single real variable. If $u_0(x)$ is bounded and of bounded variation in the sense of Tonelli-Cesari, then there exists a function $u(t,x)$ which is a weak solution of (1) in the region $t > 0$ having $u_0(x)$ as initial value. Moreover, for each fixed t, $u(t,x)$ is also bounded and of bounded variation, and $u(t,x)$ has the same upper and lower bounds as $u_0(x)$.*

Vol'pert[10] and Kruzkov[5] also proved the same result as Theorem 4.1 using the vanishing viscosity method. Kruzkov construct a theory of generalized solutions in the large of Cauchy problem for the equations

$$u_t + \sum_{i=1}^{n} \frac{d}{dx_i} f^i(t,x,u) + g(t,x,u) = 0 \tag{42}$$

in the class of bounded measurable functions. He define the generalized solution and prove existence uniqueness and stability theorems for this solution. To prove these theorem, he first consider the Cauchy's problem for the corresponding parabolic equation

$$u_t + \sum_{i=1}^{n} \frac{d}{dx_i} f^i(t,x,u) + g(t,x,u) = \epsilon \Delta u, \quad \epsilon > 0, \tag{43}$$

and derive a priori estimates

$$\int_{B(0:r)} |u^\epsilon(t, x + \Delta x) - u^\epsilon(t, x)|\, dx \leq \omega_r^x(|\Delta x|) \tag{44}$$

$$\int_{B(0;r)} |u^\epsilon(t + \Delta t, x) \quad - u^\epsilon(t, x)|\, dx \leq \omega_r^t(|\Delta t|) \tag{45}$$

$$= \text{const.} \min_{0<h\leq 1} [h + \omega_r^x(h) + \frac{\Delta t^2}{h}]$$

$$\int_{B(0;r)} |u^\epsilon(s, x) - u_0(x)|\, dx \leq \omega_r^t(s) \tag{46}$$

for any $r > 0$ and $s \in [0, T]$, where $\omega(h)$ is a modulus of continuity type functions and the constant does not dependent of ϵ. Passing to the limit as $\epsilon \to 0$, the solution u^ϵ of (43) and (2) converges to u almost everywhere.

References

1. T. Chang and L. Hsiao, *The Riemann problem and Interaction of Waves in Gas Dynamics*, Pitman Mono. and Surveys in Pure and Appl. Math. 41, Longman Sci. and Tech. New York, 1989
2. E. Conway and J. Smoller, *Global Solutions of the Cauchy Problem for Quasi-Linear First-Order Equations in Several Space Variables*, Comm. Pure Appl. Math. 19(1966), 95–105
3. C. Dafermos, *Quasilinear Hyperbolic Systems with Involutions*, Arch. Rat. Mech. Anal. 94(1986), 373–389
4. C. Dafermos, *Entropy for Hyperbolic Systems of Conservation Laws in Several Space Dimensions*, Nonlinear Evolutionary Partial Differential Equations, X. Ding and T.P. Liu, AMS/IP Studies in Adv. Math. 3, Amer. Math. Soc. New York ,1997, 27–41
5. S. N. Kruzkov, *First-Order Equations in Several Independent Variables*, Math. USSR Sb. 10(1970), 217–243
6. P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. 10(1957), 537–566

7. A. Majdar, Compressible Fluid Flow and System of Conservation laws in Several Space Variables, Applied Math. Sci. 53, Springer-Verlag, New York, 1984

8. J. Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one*, Comm. Math. Phys. 106(1986), 481–484

9. J. Smoller, Shock Waves and Reaction-Diffusion Equations, Grundl. math. Wissen. 258, Springer-Verlag, New York, 1982

10. A. I. Vol'pert, The space BV and Quasi-Linear Equations, Math. USSR Sb. 2(1967), 225–267

11. Choon Ho Lee, Personal Notes(1997)

DEPARTMENT OF MATHEMATICS
HOSEO UNIVERSITY
ASAN 336-795, KOREA
E-mail: chlee@math.hoseo.ac.kr