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FUZZY LINEARITY OF THE FUZZY INTEGRAL

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ABSTRACT. We introduce a concept of fuzzy linearity: A function $F : L^0(X) \longrightarrow \mathbb{R}$ is *fuzzy linear* if $F\ [\ (a \wedge f) \vee (b \wedge g)\ ] = [\ a \wedge F(f)\ ] \vee [\ b \wedge F(g)\ ]$ for $f, g \in L^0(X)$ and $a, b > 0$. We show that a fuzzy integral is fuzzy linear if the measure is fuzzy c-additive.

Sugeno [6] defined a fuzzy measure as a measure having the monotonicity instead of additivity, and a fuzzy integral which is an integral with respect to fuzzy measure. Batle and Trillas [1] and Dubois and Prade [2] studied the fuzzy integrals. A generalization of the fuzzy integral was introduced by Ralescu and Adams [4]. In general the fuzzy integral is not linear as consequence of the non-additivity of the fuzzy measure. In this paper, we will introduce, what we call, fuzzy linearity in which we use the supremum and the infimum instead of addition and scalar multiplication in the expression of linearity. The main result is that fuzzy linearity for the fuzzy integral holds when μ is fuzzy c-additive.

We recall some definitions and results used in this paper. Let X be a set and $\mathcal{F}$ be a σ-algebra of subsets of X. A set function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ is said to be a *fuzzy measure* if the following holds:

(1) $\mu(\emptyset) = 0$.

(2) $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subset B$ imply $\mu(A) \leq \mu(B)$

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(3) $\{A_n\} \subset \mathcal{F}$, $A_1 \subset A_2 \subset \cdots$, and $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ imply

$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

(4) $\{A_n\} \subset \mathcal{F}$, $A_1 \supset A_2 \supset \cdots, \mu(A_1) < \infty$ and $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$ imply

$$\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

The main difference between fuzzy measures and classical measures is the lack of additivity of the former. However each classical measure is a fuzzy measure.

The triple $(X, \mathcal{F}, \mu)$ is called a *fuzzy measure space.* A real-valued function $f : X \longrightarrow (-\infty, \infty)$ is $\mathcal{F}$-measurable with respect to $\mathcal{F}$ and $\mathcal{B}$ (measurable, for short, if there is no confusion likely) if $f^{-1}(B) = \{\, x \mid f(x) \in B \,\} \in \mathcal{F}$ for any $B \in \mathcal{B}$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$. Let $f : X \longrightarrow [0, +\infty)$ be a finite positive measurable function. The *fuzzy integral* is defined as :

$$⨍_A f \, d\mu = \sup_{\alpha \in [0,\infty]} [\, \alpha \wedge \mu(A \cap F_\alpha) \,],$$

where $F_\alpha = \{\, x \mid f(x) \geq \alpha \,\}$.

When $A = X$, the fuzzy integral may also be denoted by $⨍ f \, d\mu$. The fuzzy integral is also called Sugeno's integral. Properties and applications of this integral were discussed in [4], [6]. The following example shows that the fuzzy integral is not linear.

EXAMPLE 1. Define a set function μ on $\mathcal{F}$(the Borel σ-algebra of $\mathbb{R}$) by

$$\mu(A) = \begin{cases} 1 & \text{if} \quad A \cap \{0,1\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

μ is a fuzzy measure on $\mathcal{F}$. And for $f = \chi_{\{0\}}, g = \chi_{\{1\}}$ we get

$$⨍ f \, d\mu = ⨍ g \, d\mu = ⨍ (f+g) \, d\mu = 1.$$

If $(X, \mathcal{F}, \mu)$ is a fuzzy measure space, let us consider the set

$$L^0(X) = \{ f : X \longrightarrow [0,1] \mid f \text{ is measurable with respect to } \mathcal{F} \text{ and } \mathcal{B} \}$$

where $\mathcal{B}$ is the usual σ-algebra of Borel subsets of $[0,1]$.

In [3], we can find the following property.

THEOREM 2. ([3]) *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. Then the following statements are equivalent:*

(1) $\forall f, g \in L^0(X)$, $\forall a, b \in [0, \infty) : [\ a \cdot f + b \cdot g \in L^0(X) \Rightarrow ⨍ (a \cdot f + b \cdot g) \, d\mu = a \cdot ⨍ f \, d\mu + b \cdot ⨍ g \, d\mu\ ]$;

(2) *μ is a probability measure fulfilling $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{F}$.*

Theorem 2 above shows that only for very small classes of fuzzy measure, linearity for the fuzzy integral holds.

We say that a function $F : L^0(X) \longrightarrow \mathbb{R}$ is *fuzzy linear* if

$$F\ [\ (a \wedge f) \vee (b \wedge g)\ ] = [\ a \wedge F(f)\ ] \vee [\ b \wedge F(g)\ ],$$

where a and b are nonnegative constants.

Let $\mathcal{A}$ be a collection of subsets of X. A set function ν is called *fuzzy c-additive* on $\mathcal{A}$ if

$$\nu \left(\cup_{i \in I} E_i \right) = \sup_{i \in I} \ \nu(E_i)$$

for any subclass $\{E_i \mid i \in I\}$ of $\mathcal{A}$ whose union is in $\mathcal{A}$, where I is an arbitrary countable index set.

If $\mathcal{A}$ is a finite class, then the fuzzy c-additivity of ν on $\mathcal{A}$ is equivalent to the simpler requirement that

$$\nu(E_1 \cup E_2) = \nu(E_1) \vee \nu(E_2),$$

whenever $E_1 \in \mathcal{A}, E_2 \in \mathcal{A}$.

Note that the measure μ in Theorem 2 is clearly fuzzy c-additive. Now we will show that fuzzy linearity for the fuzzy integral holds when μ is fuzzy c-additive.

LEMMA 3. *Let $f, g, h \in L^0(X)$, $h(x) = f(x) \vee g(x)$, and $H_\alpha = \{ x \mid h(x) \geq \alpha \}$. Then $H_\alpha = F_\alpha \cup G_\alpha$, where $F_\alpha = \{ x \mid f(x) \geq \alpha \}$, $G_\alpha = \{ x \mid g(x) \geq \alpha \}$.*

Proof. For any $\alpha \in [0, \infty]$,

$$\begin{aligned} H_\alpha = \{ x \mid h(x) \geq \alpha \} &= \{ x \mid (f \vee g)(x) \geq \alpha \} \\ &= \{ x \mid f(x) \geq \alpha \text{ or } g(x) \geq \alpha \} \\ &= \{ x \mid f(x) \geq \alpha \} \cup \{ x \mid g(x) \geq \alpha \} \\ &= F_\alpha \cup G_\alpha \end{aligned}$$ □

THEOREM 4. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. If μ be fuzzy c-additive then*

$$⨍_A [\, (a \wedge f) \vee (b \wedge g) \,]\, d\mu = (a \wedge ⨍_A f\, d\mu) \vee (b \wedge ⨍_A g\, d\mu).$$

for any $A \in \mathcal{F}$ and $f, g \in L^0(X)$, where a and b are nonnegative constants.

Proof. We may assume that $A = X$ without loss of generality. Let $F_\alpha^* = \{ x \mid a \wedge f(x) \geq \alpha \}$ and $G_\alpha^* = \{ x \mid b \wedge g(x) \geq \alpha \}$. Using the result of Lemma 3 and the fuzzy c-additivity of μ, we have

$$\begin{aligned}
⨍[\,(a\wedge f)\vee(b\wedge g)\,]\,d\mu &= \sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\mu(\,\{\,x \mid [(a\wedge f)\vee(b\wedge g)](x)\geq\alpha\}\,)\,] \\
&= \sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\mu(F^*_\alpha\cup G^*_\alpha)\,] \\
&= \sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\{\mu(F^*_\alpha)\vee\mu(G^*_\alpha)\}\,] \\
&= \sup_{\alpha\in[0,\infty]}[\,\{\,\alpha\wedge\mu(F^*_\alpha)\,\}\vee\{\,\alpha\wedge\mu(G^*_\alpha)\,\}\,] \\
&\leq \sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\mu(F^*_\alpha)\,]\vee\sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\mu(G^*_\alpha)\,] \\
&= ⨍(a\wedge f)\,d\mu\vee⨍(b\wedge g)\,d\mu \qquad (1)
\end{aligned}$$

Since $⨍ f_1\,d\mu\leq⨍ f_2\,d\mu$ *for* $f_1\leq f_2,\ f_1,f_2\in L^0(X)$,

$$⨍(f_1\vee f_2)\,d\mu\geq⨍ f_1\,d\mu\vee⨍ f_2\,d\mu.$$

Hence

$$⨍[\,(a\wedge f)\vee(b\wedge g)\,]\,d\mu\geq⨍(a\wedge f)\,d\mu\vee⨍(b\wedge g)\,d\mu. \qquad (2)$$

By (1) and (2), we have

$$⨍[\,(a\wedge f)\vee(b\wedge g)\,]\,d\mu=(a\wedge⨍ f\,d\mu)\vee(b\wedge⨍ g\,d\mu).$$

It remains to prove that $⨍(a\wedge f)\,d\mu=a\wedge⨍ f\,d\mu$. This does not need fuzzy c-additivity of μ. Since

$$F^*_\alpha=\{\,x\mid a\geq\alpha\,\}\cap\{\,x\mid f(x)\geq\alpha\,\},$$

we have $F^*_\alpha=\phi$ if $\alpha>a$, and $F^*_\alpha=F_\alpha$ if $\alpha\leq a$. Hence

$$\begin{aligned}
⨍(a\wedge f)\,d\mu &= \sup_{\alpha\in[0,\infty]}[\,\alpha\wedge\mu(F^*_\alpha)\,] \\
&= \sup_{\alpha\in[0,a]}[\,\alpha\wedge\mu(F^*_\alpha)\,]\vee\sup_{\alpha\in(a,\infty]}[\,\alpha\wedge\mu(F^*_\alpha)\,] \\
&= \sup_{\alpha\in[0,a]}[\,\alpha\wedge\mu(F_\alpha)\,].
\end{aligned}$$

Note that

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \le a.$$

We can write

$$⨍ f\ d\mu = \sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \vee \sup_{\alpha\in(a,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ].$$

First, we consider the case

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \ge \sup_{\alpha\in(a,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ].$$

Then

$$\begin{aligned} ⨍ f\ d\mu &= \sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \\ &= ⨍ (a \wedge f)\ d\mu. \end{aligned}$$

Since

$$⨍ (a \wedge f)\ d\mu = \sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \le a,$$

$$⨍ f\ d\mu \le a.$$

Hence

$$a \wedge ⨍ f\ d\mu = ⨍ f\ d\mu = ⨍ (a \wedge f)\ d\mu.$$

Secondly, we consider the case

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] < \sup_{\alpha\in(a,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ].$$

We shall show $\sup_{\alpha\in[0,a]}[\ \alpha \wedge \mu(F_\alpha)\ ] = a$ in this case.
Since

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \le a,$$

suppose, to get the contradiction, that

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] < a.$$

Then $\mu(F_a) < a$, and

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \geq a \wedge \mu(F_a) = \mu(F_a) \geq \sup_{\alpha\in(a,\infty]} \mu(F_\alpha)$$

since $\mu(F_\alpha)$ is monotone decreasing. Hence

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \geq \sup_{\alpha\in(a,\infty]} \mu(F_\alpha). \tag{3}$$

Since

$$\mu(F_\alpha) \leq \sup_{\alpha\in(a,\infty]} \mu(F_\alpha) \leq \mu(F_a) < a < \alpha \quad \text{for} \ \ \alpha \in (a,\infty],$$

$$\mu(F_\alpha) = \alpha \wedge \mu(F_\alpha), \quad \alpha \in (a,\infty]$$

and hence

$$\sup_{\alpha\in(a,\infty]} \mu(F_\alpha) = \sup_{\alpha\in[0,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ] \tag{4}$$

From (3) and (4)

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] \geq \sup_{\alpha\in(a,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ],$$

which contradicts $\sup_{\alpha\in[0,a]}[\ \alpha \wedge \mu(F_\alpha)\ ] < \sup_{\alpha\in(a,\infty]}[\ \alpha \wedge \mu(F_\alpha)\ ]$. This shows that $\sup_{\alpha\in[0,a]}[\ \alpha \wedge \mu(F_\alpha)\ ] = a$. Since

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] < \sup_{\alpha\in(a,\infty]} [\ \alpha \wedge \mu(F_\alpha)\ ]$$

and

$$\sup_{\alpha\in[0,a]} [\ \alpha \wedge \mu(F_\alpha)\ ] = a,$$

we have

$$
\begin{aligned}
⨍ f\, d\mu &= \sup_{\alpha\in[0,a]} [\, \alpha \wedge \mu(F_\alpha)\,] \vee \sup_{\alpha\in(a,\infty]} [\, \alpha \wedge \mu(F_\alpha)\,] \\
&= \sup_{\alpha\in(a,\infty]} [\, \alpha \wedge \mu(F_\alpha)\,] \\
&> \sup_{\alpha\in[0,a]} [\, \alpha \wedge \mu(F_\alpha)\,] = a.
\end{aligned}
$$

Hence

$$⨍ f\, d\mu > a.$$

Note that

$$
\begin{aligned}
⨍ (a \wedge f)\, d\mu &= \sup_{\alpha\in[0,a]} [\, \alpha \wedge \mu(F_\alpha)\,] \\
&= a.
\end{aligned}
$$

Therefore

$$a \wedge ⨍ f\, d\mu = a = ⨍ (a \wedge f)\, d\mu.$$

Thus $⨍ (a \wedge f)\, d\mu = a \wedge ⨍ f\, d\mu$ has been proven. Hence the proof of the theorem is complete. □

Corollary 5. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. If μ is fuzzy c-additive then for any f and g in $L^0(X)$,*

$$⨍_A (f \vee g)\, d\mu = ⨍_A f\, d\mu \vee ⨍_A g\, d\mu.$$

Proof. We may assume that $A = X$ without loss of generality. Since $⨍ (a \wedge f)\, d\mu = \sup_{\alpha\in[0,a]}[\, \alpha \wedge \mu(F_\alpha)\,] \leq a$,

$$⨍ f\, d\mu = ⨍ (1 \wedge f)\, d\mu \leq 1.$$

Hence

$$\begin{aligned}\,⨍[\, f \vee g \,]\, d\mu &= ⨍[(1 \wedge f) \vee (1 \wedge g)]\, d\mu \\ &= (\, 1 \wedge ⨍ f\, d\mu \,) \vee (\, 1 \wedge ⨍ g\, d\mu \,) \\ &= ⨍ f\, d\mu \vee ⨍ g\, d\mu.\end{aligned}$$

□

COROLLARY 6. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. If μ is fuzzy c-additive, and if $⨍_A \ f\, d\mu = ⨍_A \ g\, d\mu$, then*

$$⨍_A (f \vee g)\, d\mu = ⨍_A f\, d\mu.$$

The following example shows that, in Theorem 4, Corollary 5 and Corollary 6, the fuzzy c-additivity is necessary even in the case when X is a finite set.

EXAMPLE 7. Consider the problem of evaluating a Chinese dish. We consider the taste, smell, and appearance. We denote these factors by T,S, and A, respectively; hence, $X = \{T, S, A\}$. Assume further that the set function μ is employed as a fuzzy measure: $\mu(\{T\}) = 0.2$, $\mu(\{S\}) = 0.5$, $\mu(\{A\}) = 0.1$, $\mu(\{T, S\}) = 0.7$, $\mu(\{T, A\}) = 0.8$, $\mu(\{S, A\}) = 0.9$, $\mu(X) = 1$ and $\mu(\phi) = 0$. Observe that μ is not fuzzy c-additive. Two experts are invited as an adjudicator to judge each quality factor of a particular dish. And suppose that they score the quality factors as follows:

$$f(T) = 0.4,\ f(S) = 0.6,\ f(A) = 0.1;$$

$$g(T) = 0.2,\ g(S) = 0.5,\ g(A) = 0.8.$$

respectively. Then the synthetic evaluations of the quality of this dish, F and G, are calculated as follows:

$$F = \fint f \, d\mu = 0.5, \qquad G = \fint g \, d\mu = 0.5.$$

But $\fint (f \vee g) \, d\mu = 0.6$.

REFERENCES

1. Batle, N. and Trillas, E, *Entropy and fuzzy integral*, Journal of Mathematical Analysis and Applications **69** (1979), 469-474.
2. Dubois, D. and Prade, H, *Fuzzy Sets and Systems : Theory and applications*, Academic Press, New York (1980).
3. E. P. KLEMENT and D. RALESCU, *Nonlinearity of The fuzzy Integral, Fuzzy Sets and Systems.*
4. C. V. NEGOITA and D. A. RALESCU, *Applications of Fuzzy sets to Systems Analysis*, Birkhauser, Basel (1975).
5. D.RALESCU AND G.ADAMS, *The fuzzy integral*, J.Math.Anal.Appl **75**, 562-570.
6. M. SUGENO, *Theory of fuzzy integrals and its applications*, PH.D.dissertation Tokyo Institute of Technology (1974).
7. M. SUGENO, Y. TSUKAMOTO, and T. TERANO, *Subjective evaluation of fuzzy objects*, IFAC Symp. On Stochastic Control (1974).
8. Z. Wang and G. J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York and London (1992).

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