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EXISTENCE OF EQUILIBRIA IN GENERALIZED FUZZY GAMES

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ABSTRACT. The purpose of this paper is to give a new existence theorem of equilibrium in generalized fuzzy games with uncountable number of agents.

1. Introduction

As is well-known, the theory of fuzzy sets was initiated by L. Zadeh [13] as an attempt to develop a mathematically precise framework in which to treat systems or phenomena which can not be characterized precisely. Since then, wide applications of theory of fuzzy sets have been established in the areas of pattern recognition, artificial intelligence, optimization, and decision theory [2,3,13], and useful fuzzy techniques have been developed in numerous applied mathematical fields.

On the while, the classical Arrow-Debreu result [1] on the existence of Walrasian equilibria has been generalized in many directions during the last forty years. Borglin and Keiding [4] first proved a new existence theorem for a compact abstract economy with KF-majorized preference correspondences, and following their ideas, till now there

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have been many generalizations of the existence of equilibria for generalized games (e.g. [8, 9]). In a recent paper [12], the authors first introduced the concept of a fuzzy game and proved the existence of equilibrium for 1-person fuzzy game.

In this paper, by using the continuous selection theorem, we shall prove the existence theorem of equlibrium for a generalized fuzzy game with uncountable set of agents, which generalizes Theorem 3 in [12], and generalizes in turn a number of equilibrium existence theorems in [4, 8]. Also we shall give an example of a generalized fuzzy game with countable set of agents, where our theorem can be applicable but the previous equilibrium existence theorems can not be applicable.

2. Preliminaries

We first recall the following notations and definitions. Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A. Let F be a Hausdorff topological vector space and $Y \subset F$ be a non-empty convex set. We denote by $\mathcal{F}(Y)$ the collection of all fuzzy sets over Y, and a mapping from X into $\mathcal{F}(Y)$ is called a *fuzzy mapping.* If $F : X \to \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denoted by F_x in the sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x.

A fuzzy mapping $F : X \to \mathcal{F}(Y)$ is called *convex* , if for each $x \in X$, the fuzzy set F_x is a fuzzy convex set, i.e. for any $y_1, y_2 \in Y$, $t \in [0, 1]$,

$$F_x(ty_1 + (1-t)y_2) \geq \min\{F_x(y_1), F_x(y_2)\}.$$

Let A be a fuzzy set in $\mathcal{F}(Y)$. In the sequel, we denote by

$$(A)_\alpha = \{y \in Y : A(y) > \alpha\}, \quad \alpha \in [0,1),$$

the *strong* α*-cut*, and $(A)_0$ is called the *support* of A. It is easy to see that if F is a convex fuzzy mapping, then $(F_x)_\alpha$ is convex for each $\alpha \geq 0$, $x \in X$. It should be noted here that throughout this paper, we shall use the concept of strong α-cut, which is different from [5,6,7]. Therefore we shall need some new proving method for existence theorem as in [12].

Now we introduce the following general definition of fuzzy equilibrium in mathematical economics. Let I be a (possibly uncountable) set of agents. For each $i \in I$, let X_i be a non-empty set of actions and let $X = \Pi_{i \in I} X_i$. *A generalized fuzzy game* (or *abstract fuzzy economy*) $\Gamma = (X_i, A_i, B_i, P_i, \alpha_i)_{i \in I}$ is defined as a family of ordered quintuple $(X_i, A_i, B_i, P_i, \alpha_i)$ where X_i is a non-empty topological vector space (a strategy set), $\alpha_i : X \to [0,1)$ is a fuzzy constraint function, A_i, $B_i : X \to 2^{X_i}$ are constraint correspondences and $P_i : X \to \mathcal{F}(X_i)$ is a fuzzy preference correspondence. *A fuzzy equilibrium* for Γ is a choice $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap (P_{i_{\hat{x}}})_{\alpha_i(\hat{x})} = \emptyset$. Our definitions of generalized fuzzy game and a fuzzy equilibrium are fuzzy generalizations of the standard definitions as in [4,8]. In particular, when $|I| = N$, we may call the fuzzy game Γ *N-person fuzzy game.*

It should be noted that the fuzzy preference correspondence P_i is very meaningful in real economic models. In fact, the preference sets of each individual have the characters of fuzzy behaviour under certain constraints.

The following continuous selection theorem is essential in proving our main existence result :

Lemma [9]. *Let X be a non-empty paracompact Hausdorff topological space and Y be a non-empty convex subset of a topological vector space. Suppose S, $T : X \to 2^Y$ are correspondences such that*

(1) for each $x \in X$, $co\, S(x) \subset T(x)$ and $S(x) \neq \emptyset$,

(2) for each $y \in Y$, $S^{-1}(y)$ is open in X.

Then T has a continuous selection.

3. Existence of Equilibria in Generalized Fuzzy Games

As remarked before, the classical Arrow-Debreu result [1] on the existence of Walrasian equilibria has been generalized in many directions during the last forty years, and following Borglin-Keiding's technique in [4], there have been a number of generalizations of the existence of equilibria as in [8,9]. In a recent paper [12], the authors first introduced the concept of fuzzy game, and next proved the existence of equilibrium for 1-person fuzzy game.

Now we shall prove an existence theorem of equilibrium for generalized fuzzy game which is a generalization of Theorem 3 in [12] in several aspects.

Theorem. *Let $\Gamma = (X_i, A_i, B_i, P_i, \alpha_i)_{i\in I}$ be a generalized fuzzy game where I is a (possibly uncountable) set of agents, such that for each $i \in I$,*

(1) X_i is a non-empty compact convex subset of a locally convex Hausdorff topological vector space and denote $X = \Pi_{i\in I} X_i$ and $x = (x_i)_{i\in I} \in X$;

(2) the constraint correspondences $A_i, B_i : X \to 2^{X_i}$ are such that $A_i(x) \subset B_i(x)$ and $B_i(x)$ non-empty convex for each $x \in X$;

(3) the constraint correspondence $B_i : X \to 2^{X_i}$ is upper semicontinuous, and $\alpha_i : X \to [0,1)$ is upper semicontinuous ;

(4) for each $y \in X_i$, $A_i^{-1}(y)$ ($= \{x \in X : y \in A_i(x)\}$) is open in X;

(5) the preference correspondence $P_i : X \to \mathcal{F}(X_i)$ is a convex fuzzy mapping such that the function $x \mapsto (P_{ix})(y)$, for each $x \in X$, is lower semicontinuous ;

(6) for each $x \in X$, $x_i \notin (P_{ix})_{\alpha_i(x)}$, i.e. $P_{ix}(x_i) \le \alpha_i(x)$.

Then Γ *has a fuzzy equilibrium* $\hat{x} \in X$, *i.e. for each* $i \in I$,

$$\hat{x}_i \in cl\, B_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap (P_{i\hat{x}})_{\alpha_i(\hat{x})} = \emptyset.$$

Proof. For each $i \in I$, we first define a correspondence $F_i : X \to 2^{X_i}$ by

$$F_i(x) := co\, A_i(x) \cap (P_{ix})_{\alpha_i(x)}, \quad \text{for each } x \in X.$$

Then, by the assumption (5), for each $x \in X$, $F_i(x)$ is convex. And for each $y \in X_i$,

$$\begin{aligned} F_i^{-1}(y) &= \{x \in X : y \in co\, A_i(x) \cap (P_{ix})_{\alpha_i(x)}\} \\ &= \{x \in X : y \in co\, A_i(x)\} \cap \{x \in X : (P_{ix})(y) > \alpha_i(x)\} \\ &= (co\, A_i)^{-1}(y) \cap \{x \in X : (P_{ix})(y) > \alpha_i(x)\}. \end{aligned}$$

Since α_i is upper semicontinuous and $x \mapsto P_{ix}(y)$ is lower semicontinuous, $F_i^{-1}(y)$ is an open subset of X. Therefore the domain set $U_i := \{x \in X : F_i(x) \neq \emptyset\} = \bigcup_{y \in X_i} F_i^{-1}(y)$ is an open subset of the compact set X, so that U_i is paracompact. Then the restriction $F_i|_{U_i} : U_i \to 2^{X_i}$ of F_i on the paracompact set U_i satisfies the following:

(i) for each $x \in U_i$, $F_i|_{U_i}(x)$ is non-empty convex,

(ii) for each $y \in X_i$, $(F_i|_{U_i})^{-1}(y)$ is open.

Then by Lemma, we can find a continuous selection $f_i : U_i \to X_i$ of $F_i|_{U_i}$, i.e., $f_i(x) \in F_i(x)$ for each $x \in U_i$.

Finally, we define a correspondence $F : X \to 2^X$ by $F(x) := \Pi_{i \in I} G_i(x)$, for each $x \in X$, where

$$G_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } \ x \in U_i, \\ cl\, B_i(x), & \text{if } \ x \notin U_i. \end{cases}$$

Then for each $x \in X$, $G_i(x)$ is a non-empty closed convex subset of X. In order to show that G_i is upper semicontinuous, we must show that the set $O := \{x \in X : G_i(x) \subset V\}$ is open in X for every open subset V of X_i. Since $f_i(x) \in F_i|_{U_i}(x) \subset co\, A_i(x) \subset B_i(x)$ for each $x \in U_i$, we have

$$\begin{aligned} O &= \{x \in X : G_i(x) \subset V\} \\ &= \{x \in U_i : G_i(x) \subset V\} \cup \{x \in X \setminus U_i : G_i(x) \subset V\} \\ &= \{x \in U_i : f_i(x) \in V\} \cup \{x \in X \setminus U_i : cl\, B_i(x) \subset V\} \\ &= \{x \in U_i : f_i(x) \in V\} \cup \{x \in X : cl\, B_i(x) \subset V\}. \end{aligned}$$

It follows from the upper semicontinuity of $cl\, B_i$ that the set $\{x \in X : cl\, B_i(x) \subset V\}$ is open in X and by the continuity of f_i, the set $\{x \in U_i : f_i(x) \in V\}$ is open in U_i. Hence O is open in X, and G_i is upper semicontinuous. Therefore, by Lemma 3 in [10], the correspondence $F(x) = \Pi_{i \in I}\, G_i(x)$ is also upper semicontinuous.

By applying the Fan-Glicksberg fixed point theorem to F, there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$, i.e., for each $i \in I$, $\hat{x}_i \in G_i(\hat{x})$. If $\hat{x} \in U_i$, then

$$\hat{x}_i \in f_i(\hat{x}) \in F_i(\hat{x}) = co\, A_i(\hat{x}) \cap (P_{i\hat{x}})_{\alpha_i(\hat{x})} \subset (P_{i\hat{x}})_{\alpha_i(\hat{x})},$$

which contradicts the assumption (6). Therefore for each $i \in I$, $\hat{x} \notin U_i$, so that we have $\hat{x}_i \in G_i(\hat{x}) = cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap (P_{i\hat{x}})_{\alpha_i(\hat{x})} = \emptyset$. This completes the proof. □

Remarks. (1) For each $i \in I$, when $A_i(x) = B_i(x) = X_i$ for each $x \in X$, then the assumptions (2), (3) and (4) are automatically satisfied ; and then we can obtain the existence of fuzzy maximal element for P_i.

(2) For each $i \in I$, when $P_i(x) = \emptyset$ for each $x \in X$, we can obtain a generalization of the Fan-Browder fixed point theorem as a corollary.

(3) As remarked in [12], our theorem can be generalized to non-compact generalized fuzzy games with infinitely many agents by following the method in [8]. □

Finally we shall give an example of fuzzy game with countable set of agents having fuzzy equilibrium, where our theorem can be applicable but the previous equilibrium existence theorems can not be applicable:

Example. Let $\Gamma = (X_i, A_i, B_i, P_i, \alpha_i)_{i\in\mathbb{N}}$ be a generalized fuzzy game with countable set of agents such that for each $i \in \mathbb{N}$,

(1) $X_i = [0,1]$ is a non-empty compact convex strategy set;

(2) the constraint correspondence $A_i = B_i : X \to 2^{X_i}$ is defined by

$$A_i(x) := \{y \in X_i : x_i < y < (1 - \frac{1}{i})x_i + \frac{1}{i}\}, \qquad \text{for each } x \in X;$$

(3) the fuzzy preference correspondence $P_i : X \to \mathcal{F}(X_i)$ is defined by

$$P_{ix}(y) := \begin{cases} 0, & \text{if } x_i \in [0, \frac{1}{2}],\ y \in [0,1], \\ ((1 - \frac{1}{i})x_i + \frac{1}{2i})y, & \text{if } x_i \in (\frac{1}{2}, 1],\ y \in [0,1]; \end{cases}$$

(4) the fuzzy constraint function $\alpha_i : X \to [0,1)$ is defined by

$$\alpha_i(x) := \begin{cases} 0, & \text{if } x_i \in [0, \frac{1}{2}), \\ ((1 - \frac{1}{i})x_i + \frac{1}{2i})x_i, & \text{if } x_i \in [\frac{1}{2}, 1]. \end{cases}$$

Then for each $i \in \mathbb{N}$, it is clear that A_i is upper semicontinuous such that each $A_i(x)$ is non-empty convex, and for each $y \in X_i$, $A_i^{-1}(y)$ is open. Also α_i is upper semicontinuous and for each $x \in X$, $P_{ix}(x_i) \le \alpha_i(x)$. It is easy to see that the function $x \mapsto (P_{ix})(y)$, for each $x \in X$, is lower semicontinuous. Therefore all the hypotheses of Theorem are satisfied, so that there exists an equilibrium for the fuzzy game Γ. In fact, there exists a fuzzy equilibrium $\hat{x} = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \cdots) \in X$ such that for each $i \in \mathbb{N}$,

$$\tfrac{1}{2} \in cl\, A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap (P_{i\hat{x}})_{\alpha_i(\hat{x})} = \emptyset. \qquad \square$$

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