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ON SIMPLE EXTENSIONS OF δ-FRAMES

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ABSTRACT. In this paper, we introduce sub-δ-frames and simple extensions on a δ-frame, and study their properties.

1.Introduction

The study of topological properties from a lattice-theoretic viewpoint was initiated by H. Wallman [11]. In particular, C. Ehresmann [5] and J. Bénabou [2] took the decisive step of regarding local lattices as generalized topological spaces in their own right. Such a local lattice is called a frame, a term introduced by C. H. Dowker and studied by D. Papert [4], J. R. Isbell [8], B. Banaschewski [1], P. T. Johnstone [9], Jorge Picado [10], and J. Wick Pelletier [12].

We note that continuous lattices and frames are characterized by certain distributive laws. We also note that a frame L is a complete lattice but in the theory of frames, we use only finite meets. Considering countable meets, we will get more properties of frames.

In this paper, a partially ordered set is also called a poset. If $\leq$ is a partial order on L, the smallest (largest, resp.) element of L, if it exists, is the element 0 (e, resp.) such that $0 \leq x$ ($x \leq e$, resp.) for each $x \in L$. Smallest (largest, resp.) elements are unique when they exist, by antisymmetry. We call 0 (e, resp.) the bottom (top, resp.) element of L. From now on, we denote a poset $(L, \leq)$ simply as L.

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DEFINITION 1.1. Let L be a poset. We say that L is :

(1) a *lattice* if every finite subset of L has the least upper bound and the greatest lower bound.
(2) *complete* if every subset A of L has the least upper bound and the greatest lower bound.

DEFINITION 1.2. A map $f : S \to T$ between two posets S and T is called an *isotone* if for $a \leq b$ in S, $f(a) \leq f(b)$ in T.

DEFINITION 1.3. Let S and T be posets and let $f : S \to T$ and $g : T \to S$ be isotones. Then (f, g) is said to be an *adjunction* or a *Galois connection* between S and T provided for any $x \in S$ and $y \in T$, $f(x) \leq y$ iff $x \leq g(y)$. In this case, f is said to be a *left adjoint* of g and g a *right adjoint* of f, and we write $f \dashv g$.

Every left adjoint preserves joins and every right adjoint preserves meets when they exist. Furthermore, let S be a complete lattice and T a poset, then $f : S \to T$ is a map which preserves joins iff f is an isotone and f has a right adjoint. Dually, a map $f : S \to T$ preserves meets iff f is an isotone and f has a left adjoint.

DEFINITION 1.4. Let E be a set, $A \subseteq E$ and R an equivalence relation on E. Then A is said to be *saturated* with respect to R if for each $x \in A$, $[x] \subseteq A$.

DEFINITION 1.5. ([6]) A complete lattice L is called a *frame* (or *complete Heyting algebra*) if for any $a \in L$ and $S \subseteq L$, $a \wedge (\bigvee S) = \bigvee\{ a \wedge s \ : \ s \in S \}$.

DEFINITION 1.6. ([3]) A frame L is called a *δ-frame* if for any $a \in L$ and countable subset K of L, $a \vee (\bigwedge K) = \bigwedge\{a \vee k : k \in K\}$.

DEFINITION 1.7. Let L and M be frames(δ-frames, resp.). Then a map $f : L \longrightarrow M$ is said to be :

(1) a *homomorphism*(*δ-homomorphism*, resp.) if f preserves finite (countable, resp.) meets and arbitrary joins.

(2) *dense* if $f(x) = 0$ implies $x = 0$.

Given a frame L and $x,\ y \in L$, we denote $x \to y = \bigvee\{z \in L\colon\ x \wedge z \leq y\}$, so that $x \wedge z \leq y$ iff $z \leq x \to y$.

DEFINITION 1.8. A frame homomorphism $f : L \to M$ is said to be *open* if for $x,\ y \in L$, $f(x \to y) = f(x) \to f(y)$.

DEFINITION 1.9. Let L be a quasi ordered set and $F \subseteq L$. Then we say that F is a *filter* (δ-*filter*, resp.) on L if F satisfies the following :

(1) F does not contain 0.

(2) $F = \uparrow F = \{x \in L : a \leq x$ for some $a \in F\}$.

(3) For any finite (countable, resp.) subset K of F, there is $a \in F$ such that for all $x \in K,\ a \leq x$.

DEFINITION 1.10. A filter F on a frame L is said to be *completely prime* if $\bigvee S \in F$ $(S \subseteq L)$ implies $F \bigcap S \neq \emptyset$.

DEFINITION 1.11. Let L be a frame and $M \subseteq L$. Then M is said to be a *subframe* of L if M is closed under finite meets and arbitrary joins in L.

For a set X of filters on a frame L, let $\mathcal{P}(X)$ denote the power set lattice of X and $L \times \mathcal{P}(X)$ the product frame of L and $\mathcal{P}(X)$. Then $\{(x,\ \sum) \in L \times \mathcal{P}(X)$: for any $F \in \sum,\ x \in F\}$ is a subframe of $L \times \mathcal{P}(X)$, which is denoted by $\mathcal{S}_X L$. And the restriction $s : \mathcal{S}_X L \to L$ of the first projection $Pr_1 : L \times \mathcal{P}(X) \to L$ is an onto, dense and open homomorphism. Using these notions, we define the following :

DEFINITION 1.12. ([7]) The dense onto homomorphism $s : \mathcal{S}_X L \longrightarrow L$ is called the *simple extension* of L with respect to X.

For a frame L and any $x \in L$, let $\sum_x = \{F \in X : x \in F\}$. Then the right adjoint s_* of s is given by $s_*(x) = (x,\ \sum_x)$ for any $x \in L$. For a frame L and a set X of filters on L, if $s : \mathcal{S}_X L \to L$ is the simple extension of L with respect to X and $\mathcal{B} = \{(x,\ \sum_x) : x \in L\}$, then we have that

(1) $\mathcal{B} = s_*(L)$,

(2) $\sum_A = \sum_{\downarrow A}$.

2. Simple Extensions of δ-Frames

DEFINITION 2.1. Let L be a δ-frame and $M \subseteq L$. Then M is said to be a *sub-δ-frame* of L if M is closed under countable meets and arbitrary joins in L.

PROPOSITION 2.2.

(1) If M is a sub-δ-frame of a δ-frame L, then M is also a δ-frame.

(2) Let L and M be δ-frames. Then $L \times M$ is also a δ-frame.

Proof. (1) It is trivial. (2) For any subset $\{(x_\alpha, y_\alpha) : \alpha \in \Lambda\}$ of $L \times M$ and countable subset $\{(x_i, y_i) : i \in \mathbb{N}\}$ of $L \times M$, we define

$$\bigvee_{\alpha\in\Lambda} (x_\alpha, y_\alpha) = (\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} y_\alpha), \ \bigwedge_{i\in\mathbb{N}} (x_i, y_i) = (\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in\mathbb{N}} y_i).$$

Since $L \times M$ is a frame, it is enough to show that for any $(x, y) \in L \times M$, $(x, y) \vee (\bigwedge_{i\in\mathbb{N}} (x_i, y_i)) = \bigwedge_{i\in\mathbb{N}} ((x, y) \vee (x_i, y_i))$. Since L and M are δ-frame,

$$\begin{aligned}
(x, y) \vee (\bigwedge_{i\in\mathbb{N}} (x_i, y_i)) &= (x, y) \vee (\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in\mathbb{N}} y_i) \\
&= (x \vee (\bigwedge_{i\in\mathbb{N}} x_i), y \vee (\bigwedge_{i\in\mathbb{N}} y_i)) \\
&= (\bigwedge_{i\in\mathbb{N}} (x \vee x_i), \bigwedge_{i\in\mathbb{N}} (y \vee y_i)) \\
&= \bigwedge_{i\in\mathbb{N}} (x \vee x_i, y \vee y_i) = \bigwedge_{i\in\mathbb{N}} ((x, y) \vee (x_i, y_i)).
\end{aligned}$$

Thus $L \times M$ is a δ-frame. □

Let L be a δ-frame and X a set of δ-filters on L. Then we will denote the power set lattice by $\mathcal{P}(X)$ in which the meet and the join are given by $\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$ for any set I, respectively. Then by Proposition 2.2, $L \times \mathcal{P}(X)$ is a δ-frame. Let $\mathcal{S}_X L = \{(x, \sum) \in L \times \mathcal{P}(X) :$ for any $F \in \sum, \ x \in F\}$. Using this notation, we have :

LEMMA 2.3. *$\mathcal{S}_X L$ is a sub-δ-frame of $L \times \mathcal{P}(X)$.*

Proof. Take any countable subset $\{(x_i, \sum_i) : i \in \mathbb{N}\}$ of $\mathcal{S}_X L$. Then

$$\bigwedge_{i\in\mathbb{N}} (x_i, \textstyle\sum_i) = (\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in\mathbb{N}} \sum_i) = (\bigwedge_{i\in\mathbb{N}} x_i, \bigcap_{i\in\mathbb{N}} \sum_i) \in L \times \mathcal{P}(X).$$

If $F \in \bigcap_{i\in\mathbb{N}} \sum_i$, then $F \in \sum_i$ for all $i \in \mathbb{N}$, hence $x_i \in F$ for all $i \in \mathbb{N}$. Thus $\bigwedge_{i\in\mathbb{N}} x_i \in F$, since F is a δ-filter on L. Hence $(\bigwedge_{i\in\mathbb{N}} x_i, \bigcap_{i\in\mathbb{N}} \sum_i) \in \mathcal{S}_X L$. Thus $\mathcal{S}_X L$ is a sub-δ-frame of $L \times \mathcal{P}(X)$. □

THEOREM 2.4. *Let L be a δ-frame and X a set of δ-filters in L. Then the map $s : \mathcal{S}_X L \longrightarrow L$ given by $(x, \sum) \longmapsto x$ is a dense onto open δ-homomorphism.*

Proof. For any countable subset $\{(x_i, \sum_i) : i \in \mathbb{N}\}$ of $\mathcal{S}_X L$,

$$s(\bigwedge_{i\in\mathbb{N}} (x_i, \textstyle\sum_i)) = s((\bigwedge_{i\in\mathbb{N}} x_i, \bigcap_{i\in\mathbb{N}} \sum_i)) = \bigwedge_{i\in\mathbb{N}} x_i = \bigwedge_{i\in\mathbb{N}} s(x_i, \sum_i).$$

Since the map $s : \mathcal{S}_X L \longrightarrow L$ given by $(x, \sum) \longmapsto x$ in a frame L is a dense onto open homomorphism, s is a dense onto open δ-homomorphism. □

DEFINITION 2.5. The dense onto δ-homomorphism $s : \mathcal{S}_X L \longrightarrow L$ in Theorem 2.4 is called the *simple extension* of L with respect to X.

REMARK 2.6. Let L be a δ-frame, X a set of δ-filters on L, and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. For any $x \in L$, let $\sum_x = \{F \in X : x \in F\}$. Then the following are equivalent : (1) $(x, \sum) \in \mathcal{S}_X L$. (2) $\sum \subseteq \sum_x$. (3) $(x, \sum) \leq (x, \sum_x)$.

PROPOSITION 2.7. *Let L be a δ-frame, X a set of δ-filters on L, and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Let $s_* : L \longrightarrow \mathcal{S}_X L$ be the map defined by $s_*(x) = (x, \sum_x)$ for any $x \in L$. Then s_* is a right adjoint of the simple extension s.*

Proof. Since L is a frame and s is the simple extension of L with respect to X, it is trivial. □

LEMMA 2.8. *Let L be a δ-frame, X a set of δ-filters on L, and $s : \mathcal{S}_X L \to L$ a simple extension of L with respect to X. Then for any countable subset $\{x_i : i \in \mathbb{N}\}$ of L,*

$$\sum_{\bigwedge_{i\in\mathbb{N}} x_i} = \bigcap_{i\in\mathbb{N}} \sum_{x_i} .$$

Proof. Since F is a δ-filter on L,

$$\begin{aligned} F \in \sum_{\bigwedge_{i\in\mathbb{N}} x_i} &\Longleftrightarrow \bigwedge_{i\in\mathbb{N}} x_i \in F \\ &\Longleftrightarrow x_i \in F \text{ for all } i \in \mathbb{N} \\ &\Longleftrightarrow F \in \sum_{x_i} \text{ for all } i \in \mathbb{N} \\ &\Longleftrightarrow F \in \bigcap_{i\in\mathbb{N}} \sum_{x_i} . \end{aligned}$$

□

THEOREM 2.9. *Let L and M be δ-frames, X a set of δ-filters on L and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Let $f : M \to L$ be a δ-homomorphism. Then the followings are equivalent.*

(1) *There is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$.*

(2) *There is a δ-homomorphism $T : M \to \mathcal{P}(X)$ such that*
$$T(y) \subseteq \sum_{f(y)}$$
for all $y \in M$.

Proof. (1) ⇒ (2) Put

$$T : M \xrightarrow{g} \mathcal{S}_X L \xrightarrow[\text{inclusion}]{j} L \times \mathcal{P}(X) \xrightarrow{Pr_2} \mathcal{P}(X)$$

Since g, j, and Pr_2 are δ-homomorphism, T is a δ-homomorphism. For any $y \in M$, let $g(y) = (a, \sum)$, then $g(y) = (f(y), \sum)$, for $s(g(y)) = f(y)$; hence

$$T(y) = (Pr_2 \circ j \circ g)(y) = Pr_2(g(y)) = \sum .$$

Thus by Remark 2.6.(2), $T(y) = \sum \subseteq \sum_{f(y)}$.

(2) $\Rightarrow$ (1) Define $g : M \to \mathcal{S}_X L$ by $g(y) = (f(y),\ T(y))$ for any $y \in M$. Since f and T are δ-homomorphisms, for any countable subset $\{y_i : i \in \mathbb{N}\}$ of M,

$$\begin{aligned} g(\bigwedge_{i\in\mathbb{N}} y_i) &= (f(\bigwedge_{i\in\mathbb{N}} y_i),\ T(\bigwedge_{i\in\mathbb{N}} y_i)) = (\bigwedge_{i\in\mathbb{N}} f(y_i),\ \bigwedge_{i\in\mathbb{N}} T(y_i)) \\ &= \bigwedge_{i\in\mathbb{N}} (f(y_i),\ T(y_i)) = \bigwedge_{i\in\mathbb{N}} g(y_i). \end{aligned}$$

Hence g is a δ-homomorphism. And $(s \circ g)(y) = f(y)$, for any $y \in M$. $\square$

In general, for any subset $A \subseteq L$, $\bigcup_{x\in A} \sum_x \subseteq \sum_{\bigvee A}$ but $\bigcup_{x\in A} \sum_x \neq \sum_{\bigvee A}$. For example, let $L = \{0,\ a,\ b,\ e\}$, where $0 \leq a,\ b \leq e$, and a and b are non-comparable. Let $X = \{\{e\},\ \{a,\ e\}\}$. Then $\mathcal{S}_X L$ is given by

$$\{\ (0,\ \emptyset),\ (a,\ \emptyset), (b,\ \emptyset),\ (e,\ \emptyset),$$

$$(a,\ \{\{a,\ e\}\}),\ (e,\ \{\{e\}\}),\ (e,\ \{\{a,\ e\}\}),\ (e,\ X)\ \}$$

and $\sum_a \bigcup \sum_b = \{\{a,\ e\}\} \neq X = \sum_e = \sum_{a\vee b}$. Thus $\sum_a \bigcup \sum_b \neq \sum_{a\vee b}$.

REMARK 2.10. Let L be a δ-frame and X a set of completely prime δ-filters on L. Then for $A \subseteq L$,

$$\sum_{\bigvee A} = \bigcup_{a\in A} \sum_a .$$

Proof. In general, $\bigcup_{a\in A} \sum_a \subseteq \sum_{\bigvee A}$. To show the reverse, take any $F \in \sum_{\bigvee A}$. Then $\bigvee A \in F \Longrightarrow a \in F$ for some $a \in A \Longrightarrow F \in \sum_a$ for some $a \in A \Longrightarrow F \in \bigcup_{a\in A} \sum_a$. □

THEOREM 2.11. *Let L and M be δ-frames, X a set of completely prime δ-filters on L, and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Then for any δ-homomorphism $f : M \to L$, there is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$.*

Proof. Let a map $T : M \to \mathcal{P}(X)$ define as $T(y) = \sum_{f(y)}$ for any $y \in M$. Since f is a δ-homomorphism and by Remark 2.10, for any subset $\{y_\alpha : \alpha \in \Lambda\}$ of M,

$$T(\bigvee_{\alpha\in\Lambda} y_\alpha) = \sum_{f(\bigvee_{\alpha\in\Lambda} y_\alpha)} = \sum_{\bigvee_{\alpha\in\Lambda} f(y_\alpha)} = \bigcup_{\alpha\in\Lambda} \sum_{f(y_\alpha)} = \bigvee_{\alpha\in\Lambda} T(y_\alpha).$$

And T is a δ-homomorphism such that $T(y) = \sum_{f(y)} \subseteq \sum_{f(y)}$ for all $y \in M$ by Lemma 2.8. Thus there is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$ by Theorem 2.9. □

THEOREM 2.12. *Let f be a dense δ-homomorphism from a δ-frame M to a δ-frame L. Let Y be a set of completely prime δ-filters on M which are saturated with respect to $ker(f) = \{(a,b) : f(a) = f(b)\}$. Let $X = \{[f(F)] : F \in Y\}$, where $[f(F)]$ denotes the δ-filter generated by $f(F)$, and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Then there is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$.*

Proof. It easy to show that $[f(F)]$ is a δ-filter on L, since f is a dense δ-homomorphism and F is a δ-filter. Let $T : M \to \mathcal{P}(X)$ be the map defined by $T(y) = \sum_{f(y)}$ for any $y \in M$. Since f is a δ-homomorphism and F is a δ-filter on M, T preserves countable meets by Lemma 2.8. To show that T is a δ-homomorphism, it is enough to show that for any $S \subseteq M$, $T(\bigvee S) = \bigvee_{s\in S} T(s)$. Take any subset S of

M. Then

$$\begin{aligned}[f(F)] \in \bigcup_{s\in S} \Sigma_{f(s)} &\Longrightarrow [f(F)] \in \Sigma_{f(s)} \text{ for some } s \in S \\ &\Longrightarrow f(s) \in [f(F)] \text{ for some } s \in S \\ &\Longrightarrow \bigvee_{s\in S} f(s) \in [f(F)] \\ &\Longrightarrow [f(F)] \in \Sigma_{\bigvee_{s\in S} f(s)}.\end{aligned}$$

Moreover,

$$\begin{aligned}[f(F)] \in \Sigma_{\bigvee_{s\in S} f(s)} &\Longrightarrow \bigvee_{s\in S} f(s) \in [f(F)] \\ &\Longrightarrow f(y) \leq \bigvee_{s\in S} f(s) \text{ for some } y \in F\end{aligned}$$

and hence

$$f(y) = f(y) \wedge (\bigvee_{s\in S} f(s)) = \bigvee_{s\in S} (f(y) \wedge f(s)) = f(\bigvee_{s\in S} (y \wedge s)).$$

Thus $(y, \bigvee_{s\in S} (y \wedge s)) \in ker(f)$. Since F is saturated with respect to $ker(f)$, $\bigvee_{s\in S} (y \wedge s) \in [y] \subseteq F$. Since F is a completely prime δ-filter on M, $y \wedge s \in F$ for some $s \in S$; hence $s \in F$. Thus for some $s \in S$,

$$[f(F)] \in \Sigma_{f(s)} \subseteq \bigcup_{s\in S} \Sigma_{f(s)}.$$

Therefore $\Sigma_{\bigvee_{s\in S} f(s)} \subseteq \bigcup_{s\in S} \Sigma_{f(s)}$; hence $\bigcup_{s\in S} \Sigma_{f(s)} = \Sigma_{\bigvee_{s\in S} f(s)}$. Hence $T(\bigvee_{s\in S} s) = \Sigma_{f(\bigvee_{s\in S} s)} = \Sigma_{\bigvee_{s\in S} f(s)} = \bigcup_{s\in S} \Sigma_{f(s)} = \bigvee_{s\in S} T(s)$. Thus T is a δ-homomorphism. There is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$ by Theorem 2.9. □

COROLLARY 2.13. *Let L and M be δ-frames and $f : M \to L$ a dense onto δ-homomorphism. Let Y be a set of completely prime δ-filters on M which are saturated with respect to $ker(f)$. Let $X =$*

$\{f(F) : F \in Y\}$ and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Then there is a δ-homomorphism $g : M \to \mathcal{S}_X L$ with $s \circ g = f$.

Proof. It is immediate from Theorem 2.12 and the fact that $[f(F)] = f(F)$, because f is onto. $\square$

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