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ON QS–ALGEBRAS

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ABSTRACT. In this paper, we introduce a new notion, called an QS-algebra, which is related to the areas of BCI/BCK-algebras and discuss the G-part of QS-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced a new notion, called a BH-algebra, i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (VI) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and showed that there is a maximal ideal in bounded BH-algebras. The present authors ([7]) introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In this paper we introduce a new notion, QS-algebra and discuss some properties of the G-part of QS-algebras.

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2. QS-algebras

A *Q-algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

Every Q-algebra X satisfies following condition:

(IV) $(x * (x * y)) * y = 0$

for any $x, y \in X$.

DEFINITION 2.1. A Q-algebra $(X; *, 0)$ is called a *QS-algebra* if

(V) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

For brevity we shall call X a QS-algebra unless otherwise specified. In X we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$. Note that every QS-algebra is a Q-algebra.

EXAMPLE 2.2. Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz | z \in \mathbb{Z}\}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are both Q-algebras and QS-algebras, where "$-$" is the usual subtraction of integers. Also, $(\mathbb{R}; -, 0)$ and $(\mathbb{C}; -, 0)$ are Q-algebras and QS-algebras where $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers and "$-$" is the usual subtraction of real (complex) numbers.

EXAMPLE 2.3. Let $X = \{0, 1, 2\}$ with the Cayley table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is both a Q-algebra and a QS-algebra, but not a BCH/BCI $/BCK$-algebra, since (VI) does not hold.

EXAMPLE 2.4. Let $X = \{0, 1, 2, \cdots, \omega\}$ in which $*$ is defined by

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ \omega & \text{if } y < x \text{ and } y \neq 0, \\ x & \text{if } y < x \text{ and } y = 0, \end{cases}$$

where $\leq$ is the natural ordering on X and $x < y$ denotes that $x \leq y$ and $x \neq y$. Then $(X; *, 0)$ satisfies (I), (II) and (III), but not (V) since $((2 * 1) * (2 * 0)) * (0 * 1) = \omega \neq 0$. Therefore $(X; *, 0)$ is a Q-algebra which is not a QS-algebra.

PROPOSITION 2.5. *Let X be a QS-algebra. Then for any x, y and z in X, the following hold :*

(a) $x \leq y$ *implies* $z * y \leq z * x$,
(b) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,
(c) $x * y \leq z$ *implies* $x * z \leq y$,
(d) $(x * z) * (y * z) \leq x * y$,
(e) $x \leq y$ *implies* $x * z \leq y * z$,
(f) $0 * (0 * (0 * x)) = 0 * x$.

Proof.
(a) If $x \leq y$, then $x * y = 0$. By (V), $(z * y) * (z * x) = 0$. Hence $z * y \leq z * x$.
(b) If $x \leq y$ and $y \leq z$, then by (a), $x * z \leq x * y$. By applying (II) and (V), $x * z = (x * z) * 0 = (x * z) * (x * y) = y * z = 0$. Hence $x * z = 0$ and so $x \leq z$.
(c) It follows from (III).
(d) By (I), (III) and (V),

$$((x*z)*(y*z))*(x*y) = ((x*z)*(x*y))*(y*z) = (y*z)*(y*z) = 0.$$

Thus $(x * z) * (y * z) \leq x * y$.

(e) If $x \le y$, then by (d) $(x * z) * (y * z) \le x * y = 0$, i.e., $((x * z) * (y * z)) * 0 = 0$. By applying (II), we obtain

$$(x * z) * (y * z) = 0, \text{ i.e., } x * z \le y * z.$$

(f) By (I) and (V),

$0 * (0 * (0 * x)) = (0 * 0) * (0 * (0 * x)) = (0 * x) * 0 = 0 * x.$ □

We now investigate some relations between QS-algebras and BCI/ BCK-algebras. The following theorems are easily proved, and omit the proof.

THEOREM 2.6. *Every QS-algebra X satisfying the condition* (VI) *is a BCI-algebra.*

THEOREM 2.7. *Every QS-algebra X satisfying the conditions* (VI) *and*

(VII) $\quad (x * y) * x = 0$

for any $x, y \in X$ is a BCK-algebra.

THEOREM 2.8. *Every QS-algebra X satisfying $x * (x * y) = x * y$ for all $x, y, z \in X$ is a trivial algebra.*

Proof. Putting $x = y$ in the equation $x * (x * y) = x * y$, we have $x * 0 = 0$. By (II), $x = 0$. Hence X is a trivial algebra. □

3. The G-part of QS-algebras

In this section, we investigate the properties of G-part in QS-algebras.

For any QS-algebra X, the set

$$B(X) := \{x \in X | 0 * x = 0\}$$

is called a *p-radical* of X. A QS-algebra X is said to be *p-semisimple* if $B(X) = \{0\}$.

DEFINITION 3.1. Let X be a QS-algebra. For any subset S of X, we define

$$G(S) := \{x \in S |\ 0 * x = x\}.$$

In particular, if $S = X$ then we say that $G(X)$ is the *G-part* of a QS-algebra X.

The following property is obvious.

$$G(X) \cap B(X) = \{0\}.$$

LEMMA 3.2. *If X is a QS-algebra, then $a * b = a * c$ implies $0 * b = 0 * c$, where $a, b, c \in X$.*

Proof. By (I) and (III), $(a*b)*a = (a*a)*b = 0*b$ and $(a*c)*a = (a*a)*c = 0*c$. Since $a*b = a*c$, $0*b = 0*c$. □

COROLLARY 3.3. *Let X be a QS-algebra. Then the left cancellation law holds in $G(X)$.*

Proof. Let $a, b, c \in G(X)$ with $a * b = a * c$. By Lemma 3.2, $a * b = a * c$ implies $0 * b = 0 * c$. Since $b, c \in G(X)$, $b = c$. □

DEFINITION 3.4. A QS-algebra X satisfying

$$(x * y) * (z * u) = (x * z) * (y * u)$$

for any x, y, z and $u \in X$ is called a *medial* QS-algebra.

PROPOSITION 3.5. *In a medial QS-algebra, the following identity holds:*

$$x * (x * y) = y$$

for any $x, y \in X$.

Proof. Since X is medial, by applying (I), (II) and (V), $x*(x*y) = (x*0)*(x*y) = (x*x)*(0*y) = 0*(0*y) = (0*0)*(0*y) = y*0 = y$. □

THEOREM 3.6. *In a QS-algebra X, the following are equivalent: for any x, y, z and $u \in X$,*

(a) $(x * y) * (z * u) = (x * z) * (y * u)$,

(b) $x * (y * z) = (x * y) * (0 * z)$.

Proof. (a) $\Longrightarrow$ (b): Putting $z := 0$ and $u := z$ in (a), we obtain

$$(x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z).$$

(b) $\Longrightarrow$ (a): By (III), we have $(x * y) * (z * u) = (x * (z * u)) * y = ((x * z) * (0 * u)) * y = ((x * z) * y) * (0 * u) = (x * z) * (y * u)$. This completes the proof. □

Let X and Y be QS-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if

$$f(x * y) = f(x) * f(y), \ \forall x, y \in X.$$

It is easy to see that the set $Hom(X)$ of all homomorphisms of X is a medial QS-algebra, whenever X is a medial QS-algebra.

LEMMA 3.7. *Let X be a medial QS-algebra. Then the right cancellation law holds in $G(X)$.*

Proof. Let $a, b, x \in G(X)$ with $a*x = b*x$. Then $x*y = (0*x)*y = (0 * y) * x = y * x$ for any $y \in G(X)$. By Proposition 3.5,

$$a = x * (x * a) = x * (a * x) = x * (b * x) = x * (x * b) = b.$$

□

DEFINITION 3.8. Let X be a QS-algebra and $I(\neq \emptyset) \subseteq X$. I is called an *ideal* of X if it satisfies: for all $x, y, z \in X$,

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and X are ideals of X. We shall call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I said to be *proper* if $I \neq X$.

THEOREM 3.9. *In a medial QS-algebra X, $G(X)$ is an ideal of X.*

Proof. For all $x \in G(X)$, $0 * x = x \in G(X)$. Hence $0 \in G(X)$. Next, if $x * y \in G(X), y \in G(X)$, then $0 * y = y$ and $0 * (x * y) = x * y$. Hence $x * (0 * y) = x * y = 0 * (x * y) = (0 * 0) * (x * y) = (0 * x) * (0 * y)$, since X is medial. By Lemma 3.7, we obtain $x = 0 * x$ and hence $x \in G(X)$. This means that $G(X)$ is an ideal of X, completing the proof. □

A QS-algebra X is said to be *associative* if

(VIII) $(x * y) * z = x * (y * z)$.

If X is an associative QS-algebra, then for any $x \in B(X)$,

$$0 = (x * x) * x = x * (x * x) = x * 0 = x.$$

Thus $B(X)$ is a zero ideal, i.e., $B(X) = \{0\}$. Hence any associative QS-algebra X is p-semisimple.

Let X be an associative QS-algebra and $x, y \in G(X)$. Then

$$0 * (x * y) = 0 * ((x * 0) * y) = 0 * (x * (0 * y)) = (0 * x) * (0 * y) = x * y.$$

Hence $x * y \in G(X)$, i.e., $G(X)$ is closed under "$*$". For any $x \in G(X)$, we have $0 * x = x$. By (II), $x * 0 = x$ holds in a QS-algebra X. Therefore $0 * x = x * 0 = x$ in the G-part $G(X)$ of an associative QS-algebra X. This means that $(G(X); *)$ is a monoid. Moreover, $x * x = 0$ shows that x has an inverse and x is an involution. Hence we have the following:

THEOREM 3.10. *The G-part $(G(X); *)$ of an associative QS-algebra X is a group in which every element is an involution.*

PROPOSITION 3.11. *An associative QS-algebra X satisfying $0 * x = x$ for any $x \in X$ is commutative, i.e., $x * y = y * x$ for any $x, y \in X$.*

Proof. For any $x, y \in X$,

$$\begin{aligned} y * x &= 0 * (y * x) \\ &= 0 * ((0 * y) * (0 * x)) \\ &= 0 * ((0 * (0 * x)) * y) && [\text{ by (III)}] \\ &= (0 * (0 * (0 * x))) * y && [\text{ by (VIII)}] \\ &= (0 * x) * y && [\text{ by Proposition 2.5-(f)}] \\ &= x * y, \end{aligned}$$

proving the proposition. □

COROLLARY 3.12. *The G-part $(G(X); *)$ of an associative QS-algebra X is an abelian group in which every element is an involution.*

Proof. It follows immediately from Proposition 3.11. and definition of the G-part. □

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