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SOME PROPERTIES ON FINSLER SPACES WITH A QUARTIC METRIC

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ABSTRACT. The purpose of the present paper is devoted to a study of some properties on spaces with a quartic metric from the standpoint of Finsler geometry.

0. Introduction

The so-called *quartic metric* on a differentiable manifold with the local coordinates x^i is defined by

$$L(x,y) = \left(a_{hijk}(x)y^h y^i y^j y^k\right)^{1/4} \quad (y^i = \dot{x}^i), \tag{0.1}$$

where $a_{hijk}(x)$ are components of a symmetric tensor field of $(0,4)$-type, depending on the position x alone, and a Finsler space with a quartic metric is called the *quartic Finsler space.*

We have had few papers studying quartic Finsler spaces ([3], [11], [12], [13]) although there are various papers on the geometry of spaces with a quartic metric as a generalization of the Euclidean or Riemannian geometry. The purpose of the present paper is to study spaces with a quartic metric from the standpoint of Finsler geometry.

The first section is devoted to developing a fundamental treatment of quartic Finsler spaces and a characterization of such spaces is given in terms of well-known tensors in Finsler geometry. The second section

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is devoted to finding Berwald spaces and Landsberg spaces among quartic Finsler spaces. In the third section a characteristic Finsler connection is defined in a quartic Finsler space from the standpoint of the generalized metric space due to A. Moór.

1. Characterization of quartic metrics

We consider an n-dimensional Finsler space F^n with a quartic metric $L(x,y)$ defined by (0.1). Putting

$$(1.1)\qquad \begin{aligned} &La_{ijk}(x,y) = a_{ijkr}y^r, \quad L^2a_{ij}(x,y) = a_{ijrs}y^r y^s,\\ &L^3a_i(x,y) = a_{irst}y^r y^s y^t, \end{aligned}$$

the normalized supporting element $l_i = \dot{\partial}_i L$, the angular metric tensor $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ and the fundamental tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j L^2/2 = h_{ij} + l_i l_j$ are respectively given by the equations

$$(1.2)\qquad \begin{aligned} &\text{a) } l_i = a_i, \qquad \text{b) } h_{ij} = 3(a_{ij} - a_i a_j),\\ &\text{c) } g_{ij} = 3a_{ij} - 2a_i a_j. \end{aligned}$$

The problem appearing first in treating special Finsler metrics of an interesting concrete form is to find the inverse metric (g^{ij}) of the metric (g_{ij}). In case of a quartic metric the problem is easy as follows:

DEFINITION. A quartic Finsler space or some domain of the space is called *regular*, if the intrinsic metric tensor $a_{ij}(x,y)$ has non-vanishing determinant.

Then, by the inverse matrix (a^{ij}) of (a_{ij}) the contravariant components g^{ij} of the fundamental tensor are written as $g^{ij} = [a^{ij} + 2a^i a^j/(3 - 2a^2)]/3$, as it is easily verified ([5]), where $a^i = a^{ir}a_r$ and $a^2 = a^i a_i$. It follows from this and (1.2)a) that $l^i = y^i/L = a^i/(3 - 2a^2)$ and $l^i l_i = 1 = a^2/(3 - 2a^2)$. Thus $a^2 = 1$ is derived, so g^{ij} are of the following simple form:

$$(1.3)\qquad g^{ij} = (a^{ij} + 2a^i a^j)/3.$$

As a consequence, in a regular Finsler space with a quartic metric, the usual processes of raising and lowering of indices are introduced.

Let us return to our subject. It is easy to show

$$\dot{\partial}_j a_i = 3(a_{ij} - a_i a_j)/L, \quad \dot{\partial}_k a_{ij} = 2(a_{ijk} - a_{ij} a_k)/L.$$

Therefore it follows from (1.2)c) that the covariant components $C_{ijk} = \dot{\partial}_k g_{ij}/2$ of the $(h)hv$-torsion tensor of the Cartan connection $C\Gamma$ are written as

$$LC_{ijk} = 3(a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_i a_j a_k). \tag{1.4}$$

It is well-known that a Finsler space is Riemannian, iff $C_{ijk} = 0$. This characterization of Riemannian metric is nothing but the equation $\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2 = 0$. A cubic metric $L(x, y)$ is characterized by the equation $\dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^3 = 0$. Similarly a quartic metric $L(x, y)$ is characterized by the equation $\dot{\partial}_l \dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^4 = 0$. Making use of the well-known T-tensor T_{hijk}, we get generally

$$\begin{aligned} \dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k \dot{\partial}_l L^4 &= 16 T_{hijkl} + 8\{l_h T_{ijkl} + (5)\} \\ &+ 8\{LC_h{}^r{}_i T_{rjkl} + 2C_{hij} h_{kl} + (10)\}, \end{aligned} \tag{1.5}$$

where by the abbreviation $\{\cdots + (\cdots)\}$ we denote the cyclic permutation of indices and summation such that $\{\cdots + (\cdots)\}$ becomes completely symmetric in all the indices.

Consequently the characterization theorem of quartic metric is established as follows:

THEOREM 1. *A Finsler space is one with a quartic metric, if and only if the equation*

$$2T_{hijkl} + \{l_h T_{ijkl} + (5)\} + \{LC_h{}^r{}_i T_{rjkl} + 2C_{hij} h_{kl} + (10)\} = 0$$

holds identically.

REMARK. The importance of the T-tensor has been recently noticed ([2]). It seems that the study of the T-tensor must be further promoted on account of Theorem 1.

REMARK. It is noteworthy that the rather complicated equation in Theorem 1 means the simple differential equation $\dot{\partial}_h\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\dot{\partial}_l L^4 = 0$ for the fundamental function $L(x,y)$, similarly to $C_{ijk} = \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2/2$ in Riemannian case. In connection with this fact, we recall the characteristic equation

$$C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1) \tag{1.6}$$

for a *C-reducible Finsler space*; it is written in tensors derived from $L(x,y)$ by the differentiation with respect to y^i alone, i.e., $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ and $C_i = \dot{\partial}_i(\log\sqrt{g})$. It seems to us that (1.6) is apparently simpler than the equation in Theorem 1, but only two particular solutions $L = \alpha + \beta$ (Randers metric) and $\alpha^2\beta$ (Kropina metric) are known in this stage ([5]).

2. Certain important tensors of quartic Finsler spaces

It follows first from (1.3) and (1.4) that the components $C_j{}^i{}_k$ of the $(h)hv$-torsion tensor of $C\Gamma$ are given by

$$LC_j{}^i{}_k = a_j{}^i{}_k - \delta^i_j a_k - \delta^i_k a_j + a^i(2a_j a_k - a_{jk}), \tag{2.1}$$

where we put $a_j{}^i{}_k = a^{ir}a_{jrk}$. Hence the so-called torsion vector C_i is given by

$$LC_i = a_i{}^r{}_r - na_i. \tag{2.2}$$

From (2.1) the v-curvature tensor S_{hijk} of $C\Gamma$ is written in the form

$$\begin{aligned}\frac{1}{3}L^2 S_{hijk} =& a_i{}^r{}_j a_{rhk} - a_i{}^r{}_k a_{rhj} - (a_{ij}a_{hk} - a_{ik}a_{hj}) \\ &+ (a_{ij}a_h a_k + a_{hk}a_i a_j - a_{ik}a_h a_j - a_{hj}a_i a_k).\end{aligned} \tag{2.3}$$

Next, the h- and v-covariant derivatives $X_{i|j}$, $X_i|_j$ of a covariant vector field X_i with respect to the Cartan connection $C\Gamma$ are defined by

$$X_{i|j} = \partial_j X_i - \dot{\partial}_r X_i N^r{}_j - X_r F_i{}^r{}_j,$$
$$X_i|_j = \dot{\partial}_j X_i - X_r C_i{}^r{}_j,$$

where $(F_j{}^i{}_k, N^i{}_j(= F_0{}^i{}_j), C_j{}^r{}_k)$ are connection coefficients of $C\Gamma$ and suffix 0 means the contraction by the supporting element y^i.

As to a Finsler space with a quartic metric (0,1), it follows first from (1.2) a), c) that

$$a_{i|j} = 0, \quad a_{ij|k} = 0, \tag{2.4}$$

because of $l_{i|j} = 0$ and $g_{ij|k} = 0$. These are remarkable identities, as it will be seen in the following. Then the h-covariant differentiation of (1.4) leads us to the simple equation

$$LC_{ijk|l} = 3a_{ijk|l}. \tag{2.5}$$

Therefore the $(v)hv$-torsion tensor P_{ijk} given by ([8], 17.23) is written as

$$LP_{ijk} = LC_{ijk|0} = 3a_{ijk|0}. \tag{2.6}$$

As a consequence of these equations, the equation ([8], 17.23) expressing the hv-curvature tensor P_{hijk} yields

$$\begin{aligned}\frac{1}{3}L^2 P_{hijk} =& L(a_{ijk|h} - a_{hjk|i}) - (a_i{}^r{}_j a_{rhk|0} - a_h{}^r{}_j a_{rik|0}) \\ &+ (a_i a_{hjk|0} - a_h a_{ijk|0}).\end{aligned} \tag{2.7}$$

DEFINITION. (1) A Finsler space is called a *Berwald space* (or affinely connected space), if the tensor $C_{ijk|l}$ vanishes identically.

(2) A Finsler space is called a *Landsberg space*, if the $(0)hv$-torsion tensor P_{ijk} vanishes identically.

It is noted that the condition $P_{ijk} = 0$ is equivalent to $P_{hijk} = 0$. From (2.5) and (2.6) we obtain immediately

THEOREM 2. *A quartic Finsler space is a Berwald space (resp. Landsberg space), if and only if the tensor $a_{ijk|l}$ (resp. $a_{ijk|0}$) vanishes identically, where the h-covariant differentiation is the one with respect to the Cartan connection.*

3. A characteristic Finsler connection in a quartic Finsler space

First of all we remember equation c) in (1.2) giving the fundamental tensor g_{ij} of a quartic Finsler space F^n. The tensor is different from the intrinsic metric tensor a_{ij} in a regular F^n. Nevertheless we have

$$L^2(x,y) = g_{ij}(x,y)y^iy^j = a_{ij}(x,y)y^iy^i. \tag{3.1}$$

This is a very interesting equation; F^n is regarded as a *generalized metric space of line-element* in Moór's sense [10], because there is generally no such a function $M(x,y)$ that a_{ij} is given by $a_{ij} = \dot{\partial}_i\dot{\partial}_j M^2/2$. A. Moór has developed various interesting results on the geometry of generalized metric space of line-element.

In viewpoint of (3.1) it seems natural to us to consider the problem determining a Finsler connection based on the intrinsic metric tensor $a_{ij}(x,y)$.

THEOREM 3. *In a regular quartic Finsler space F^n a Finsler connection $^*C\Gamma = (^*F_j{}^i{}_k, {}^*N^i{}_j, {}^*C_j{}^i{}_k)$ is uniquely determined from the intrinsic metric tensor $a_{ij}(x,y)$ by the following five axioms:*

(1) *It is h-metrical :* $a_{ij\overset{*}{|}k} = 0$.

(2) *It is v-metrical :* $a_{ij}\overset{*}{|}_k = 0$.

(3) *It is h-symmetric :* $^*T_j{}^i{}_k = {}^*F_j{}^i{}_k - {}^*F_k{}^i{}_j = 0$.

(4) *It is v-symmetric :* $^*S_j{}^i{}_k = {}^*C_j{}^i{}_k - {}^*C_k{}^i{}_j = 0$.

(5) *Its deflection tensor vanishes :* $y^{i*}{}_{|j} = {}^*N^i{}_j - {}^*F_0{}^i{}_j = 0$,

where $\overset{*}{|}$ *and* $\overset{*}{|}$ *denote respectively the h- and v-covariant differentiations with respect to* $^*C\Gamma$. *Then the connection coefficients* $^*F_j{}^i{}_k$

and ${}^*N^i{}_j$ *coincide with* $F_j{}^i{}_k$ *and* $N^i{}_j$ *respectively and* ${}^*C_j{}^i{}_k = C_j{}^i{}_k + l^i h_{jk}/(2L)$, *where* $C\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ *is the Cartan connection.*

REMARK. In Theorem 3 a Finsler connection is the concept given in [8 §9]. It is noteworthy that the above system of axioms is similar to the one for $C\Gamma$. The proof will be done also similar to the case of $C\Gamma$. It is, $\dot{\partial}_k g_{ij} = \dot{\partial}_j g_{ik}$ are full used, but for the intrinsic metric tensor a_{ij} such identities do not hold except $(\dot{\partial}_k a_{ij})y^k = 0$. We shall show another proof in the following.

Proof. The axioms (2) and (4) lead us immediately to

$$ {}^*C_j{}^i{}_k = a^{ir}(\dot{\partial}_k a_{jr} + \dot{\partial}_j a_{kr} - \dot{\partial}_r a_{jk})/2, \tag{3.2}$$

that is, the coefficients ${}^*C_j{}^i{}_k$ of the v-covariant differentiation are Christoffel symbols constructed from $a_{ij}(x,y)$ with respect to y^i. Substitution of (1.4) in $\dot{\partial}_k a_{ij} = 2(a_{ijk} - a_{ij}a_k)/L$ yields

$$\dot{\partial}_k a_{ij} = \frac{2}{3}\{C_{ijk} + (h_{jk}l_i + h_{ki}l_j)/L\}. \tag{3.3}$$

Thus (3.2) and (3.3) give the relation

$$ {}^*C_j{}^i{}_k = C_j{}^i{}_k + \frac{2}{3L}h_{jk}l^i. \tag{3.4}$$

Secondly we consider the difference $D_j{}^i{}_k = {}^*F_j{}^i{}_k - F_j{}^i{}_k$. Then the axiom (3) means $D_j{}^i{}_k = D_k{}^i{}_j$ and (5) does $D_0{}^i{}_k = {}^*N^i{}_k - N^i{}_k$. Pay attention to the remarkable equation, the second of (2.4); in virtue of it the axiom (1) is written in the simple form

$$(\dot{\partial}_r a_{ij})D_0{}^r{}_k + D_{ijk} + D_{jik} = 0, \tag{3.5}$$

where $D_{ijk} = a_{jr}D_i{}^r{}_k$. By the Christoffel process ([8], p.44) and (3.3) we derive from (3.5)

$$\begin{aligned} &D_{ijk} + C_i{}^r{}_j D_{0rk} + C_j{}^r{}_k D_{0ri} - C_k{}^r{}_i D_{0rj} \\ &+ \frac{1}{L}[l_i(D_{0jk} - D_{0kj}) + l_k(D_{0ji} - D_{0ij}) + l_j(D_{0ik} + D_{0ki})] \\ &- \frac{2}{L^2}(l_i l_j D_{00k} + l_j l_k D_{00i} - l_k l_i D_{00j}) = 0. \end{aligned} \tag{3.6}$$

Contraction of (3.6) by y^i yields

$$(3.7)\quad \begin{aligned} &D_{0jk} + C_j{}^r{}_k D_{0r0} - \frac{2}{L^2} l_j l_k D_{000} + (D_{0jk} - D_{0kj}) \\ &+ \frac{1}{L}\{l_j(D_{0k0} - D_{00k}) + l_k(D_{0j0} + D_{00j})\} = 0. \end{aligned}$$

$D_{0j0} = 0$ is easily obtained by contraction of (3.7) by y^k, hence (4.7) is reduced to

$$(3.8)\quad D_{0jk} + (D_{0jk} - D_{0kj}) - \frac{1}{L}(l_j D_{00k} - l_k D_{00j}) = 0.$$

Further contraction of the above by y^j gives $D_{00k} = 0$ and we have $D_{0jk} = 0$ easily. Consequently (3.6) yields the conclusion $D_{ijk} = 0$. □

REMARK. The concept of indicatrization is recently introduced ([8], Definition 31.3). It is easily verified by (3.4) that $C_j{}^i{}_k$ is the indicatrized tensor of ${}^*C_j{}^i{}_k$.

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