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ON ANTI FUZZY PRIME IDEALS IN BCK-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of anti fuzzy prime ideals in a commutative BCK-algebra and obtain some properties of it.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [10]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, etc. In 1991, Xi [9] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and Iséki [5]. He got some interesting results. In [2], Biswas introduced the concept of anti fuzzy subgroups of groups. Recently, S. M. Hong and Y. B. Jun [4], modifying Biswas' idea, apply the concept to BCK-algebras. So, they defined the notion of anti fuzzy ideals of BCK-algebras and obtained some useful results on it.

In this paper, we introduce the notion of anti fuzzy prime ideals of commutative BCK-algebras. We show that every anti fuzzy prime ideal of a commutative BCK-algebra is an anti fuzzy ideal of X. We also prove that if a fuzzy subset is an anti fuzzy prime ideal of X then so is the fuzzification of its lower level cut.

2. Preliminaries

DEFINITION 2.1. An algebra $(X, *, 0)$ of type (2,0) is called a BCK-algebra if for all $x, y, z \in X$ the following conditions hold:

(a) $((x * y) * (x * z)) * (z * y) = 0$

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(b) $(x * (x * y)) * y = 0$
(c) $x * x = 0$
(d) $0 * x = 0$
(e) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $x * y = 0$. This ordering is called BCK-ordering.

PROPOSITION 2.1. *In any BCK-algebra X, the following holds: for all $x, y, z \in X$,*
(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x * y \leq x$,
(4) $(x * y) * z \leq (x * z) * (y * z)$,
(5) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$.

A BCK-algebra X satisfying the identity $x*(x*y) = y*(y*x)$ is said to be *commutative*. We denote $x * (x * y)$ by $y \wedge x$. In a commutative BCK-algebra, it is well-known that $x \wedge y$ is the greatest lower bound of x and y.

DEFINITION 2.2. [6] A non-empty subset I of a BCK-algebra X is called an *ideal* of X if
(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply $x \in I$.

DEFINITION 2.3. [3] Let S be a non-empty set. A *fuzzy* subset μ of S is a function $S \to [0, 1]$.

DEFINITION 2.4. [3] Let μ be a fuzzy subset of S. Then for $t \in [0, 1]$, the *level subset* of μ is the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$.

DEFINITION 2.5. [9] Let X be a BCK-algebra. A fuzzy subset μ of X is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

DEFINITION 2.6. [9] Let X be a BCK-algebras. A fuzzy subset μ of X is called a *fuzzy ideal* of X if, for $x, y \in X$,

(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

DEFINITION 2.7. [4] A fuzzy subset μ of a BCK-algebra X is called an *anti fuzzy subalgebra* of X if

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

PROPOSITION 2.2. [4] *Every anti fuzzy subalgebra μ of a BCK-algebra X satisfies $\mu(0) \leq \mu(x)$ for all $x \in X$.*

DEFINITION 2.8. [4] A fuzzy subset μ of a BCK-algebra X is called an *anti fuzzy ideal* of X if

(1) $\mu(0) \leq \mu(x)$,
(2) $\mu(x) \leq \max\{\mu(x * y), \mu(y)\}$,

for all $x, y \in X$.

THEOREM 2.3. [4] *Every anti fuzzy ideal μ of a BCK-algebra X is an anti fuzzy subalgebra of X.*

REMARK 2.1. An anti fuzzy subalgebra of a BCK-algebra X need not be an anti fuzzy ideal of X, in general.

DEFINITION 2.9. [1] An ideal I of a commutative BCK-algebra X is said to be *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

DEFINITION 2.10. [7] A non-constant fuzzy ideal μ of a commutative BCK-algebra X is said to be *fuzzy prime* if

$$\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

THEOREM 2.4. [7] *If μ is a fuzzy prime ideal of a commutative BCK-algebra, then the set*

$$X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$$

is a prime ideal of X.

DEFINITION 2.11. [4] Let μ be a fuzzy subset of a BCK-algebra X. Then for $t \in [0,1]$, the set

$$\mu^t = \{x \in X \mid \mu(x) \leq t\}$$

is called the *lower t-level cut* of μ.

Note that $\mu^1 = X$ and $\mu^t \cup \mu_t = X$ for $t \in [0,1]$. Clearly, $\mu^s \subset \mu^t$ whenever $s < t$.

THEOREM 2.5. [4] *Let μ be a fuzzy subset of a BCK-algebra X. Then μ is an anti fuzzy ideal of X if and only if for each t-level cut μ^t is an ideal of X.*

3. Anti fuzzy prime ideals

DEFINITION 3.1. A non-constant anti fuzzy ideal μ of a commutative BCK-algebra X is said to be *anti fuzzy prime* if

$$\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

EXAMPLE 3.1. Let $X = \{0, x, y, z\}$ with the Cayley table as follows:

$*$	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	x	0	0
z	z	y	x	0

It is easy to verify that$(X, *, 0)$ is a commutative BCK-algebra. Define a fuzzy subset $\mu : X \to [0,1]$ by $\mu(0) = 0, \mu(x) = 0.1, \mu(y) = 0.5$ and $\mu(z) = 1$. Then it is an anti fuzzy prime ideal of X.

PROPOSITION 3.1. *Every anti fuzzy prime ideal μ of a BCK-algebra X is an anti fuzzy ideal of X.*

Proof. It is straightforward. □

REMARK 3.1. An anti fuzzy ideal of a commutative BCK-algebra X need not be an anti fuzzy prime ideal of X as shown in the following example.

EXAMPLE 3.2. Let $X = \{0, x, y, z\}$ be a BCK-algebra with Cayley table as follows:

$*$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

Then the BCK-algebra X is commutative. Define $\mu : X \to [0, 1]$ by $\mu(0) = 0, \mu(x) = \mu(y) = 0.5$ and $\mu(z) = 1$. Routine calculations give that μ is an anti fuzzy ideal of X but not anti fuzzy prime ideal of X.

THEOREM 3.2. *Let μ be an anti fuzzy prime ideal of a commutative BCK-algebra X. Then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$ is a prime ideal of X.*

Proof. The fact that X_μ is an ideal follows from Theorem 2.17. Let $x, y \in X$ be such that $x \wedge y \in X_\mu$. Then $\mu(0) = \mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\} = \mu(x)$ or $\mu(y)$. It follows from (1) of Definition 2.10 that either $\mu(0) = \mu(x)$ or $\mu(0) = \mu(y)$. Thus either $x \in X_\mu$ or $y \in X_\mu$. This completes the proof. □

COROLLARY 3.3. *If μ is an anti fuzzy prime ideal of a commutative BCK-algebra X, then the set $P = \{x \in X \mid \mu(x) = 0\}$ is either empty or a prime ideal of X.*

LEMMA 3.4. *Let I be a prime ideal of a commutative BCK-algebra X and let α and β be elements of $(0, 1]$ with $\alpha < \beta$. Then the fuzzy* *subset* $\mu : X \to [0,1]$ *defined by*

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise} \end{cases}$$

is an anti fuzzy ideal of X.

Proof. Since $0 \in I$, we have $\mu(0) = \alpha \leq \mu(x)$ for all $x \in X$. Suppose that there exist $x, y \in X$ such that $\mu(x) > \max\{\mu(x * y), \mu(y)\}$. Then $\mu(x) = \beta$ and $\max\{\mu(x * y), \mu(y)\} = \alpha$. Hence $\mu(x * y) = \alpha$ and $\mu(y) = \alpha$. It follows that $x * y \in I$ and $y \in I$. Since I is an ideal, we have $x \in I$. This is a contradiction, ending the proof. □

THEOREM 3.5. *Let* P *be a prime ideal of a commutative BCK-algebra* X *and let* $\alpha \in (0, 1]$. *If* μ *is a fuzzy subset in* X *defined by*

$$\mu(x) = \begin{cases} 0 & \text{if } x \in P, \\ \alpha & \text{otherwise,} \end{cases}$$

then μ *is an anti fuzzy prime ideal of* X.

Proof. By Lemma 3.8, we know that μ is a non-constant anti fuzzy ideal of X. Let $x, y \in X$. Assume that $x \wedge y \in P$. Then either $x \in P$ or $y \in P$. Hence $\mu(x \wedge y) = 0 = \min\{\mu(x), \mu(y)\}$. If $x \wedge y \notin P$, then

$$\mu(x \wedge y) = \alpha \geq \min\{\mu(x), \mu(y)\}.$$

Thus μ is an anti fuzzy prime ideal of X. This completes the proof. □

THEOREM 3.6. *Let* P *be an ideal of a commutative BCK-algebra* X. *Then* P *is a prime ideal if and only if the complement* χ_P^c *of the characteristic function is an anti fuzzy prime ideal.*

Proof. The necessity is clear. Suppose that $x \wedge y \in P$ and $x \notin P$, for $x, y \in X$. Then

$$\begin{aligned} 0 &= \chi_P^c(x \wedge y) = 1 - \chi_P(x \wedge y) \\ &\geq 1 - \max\{\chi_P(x), \chi_P(y)\} = \min\{1 - \chi_P(x), 1 - \chi_P(y)\} \\ &= \min\{\chi_P^c(x), \chi_P^c(y)\} = \chi_P^c(y). \end{aligned}$$

It follows that $\chi_P^c(y) = 0$, so that $\chi_P(y) = 1$. Thus $y \in P$, and hence P is a prime ideal of X. This completes the proof. □

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