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DYNAMICAL PROPERTIES ON THE ITERATION OF *CF*-FUNCTIONS

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ABSTRACT. The purpose of this paper is to show that if the Fatou set $F(f)$ of a CF-meromorphic function f has two completely invariant components, then they are the only components of $F(f)$ and that the Julia set of the entire transcendental function $E_\lambda(z) = \lambda e^z$ for $0 < \lambda < \frac{1}{e}$ contains a Cantor bouquet by employing the Devaney and Tangerman's theorem[10].

1. Introduction

The Fatou set $F(R)$ of a rational function R has at most two completely invariant components, and if the Fatou set $F(R)$ has two completely invariant components, then they are the only components of $F(R)$ [5, 15].

For any entire transcendental function, there is at most one completely invariant component [2] and if the completely invariant component exists, then it is the only component of the Fatou set [11]. But it was conjectured by W. Bergweiler [6] that if a meromorphic function f has two completely invariant components of $F(f)$, then they are the only components of $F(f)$ under the assumption that the Fatou set $F(f)$ of a meromorphic function f has at most two completely invariant components. In fact, Baker, Kotus and Lü [4] proved that a CF-meromorphic function has at most two completely invariant components.

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In this paper, we study some properties of the completely invariant components of the Fatou set of a *CF*-meromorphic function and of the Julia set of an entire transcendental function. That is, in Theorem 3.9, we give a partial answer to the above question. Also, in Theorem 4.5, we prove that the Julia set of the entire transcendental function $E_\lambda(z) = \lambda e^z$ for $0 < \lambda < \frac{1}{e}$ contains a Cantor bouquet [8, 9, 10] by employing the Devaney and Tangerman's theorem [10].

2. Preliminaries

Let f be a map of $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}$, where $\overline{\mathbb{C}}$ denotes the complex sphere $\mathbb{C} \cup \{\infty\}$. We denote by f^n the n-th iterate of f, that is, $f^0(z) = z$ and $f^n(z) = f(f^{n-1}(z))$ for $n \in \mathbb{N}$. The maximal open set $F(f)$ on which the family $\{f^n : n \geq 1\}$ of iterates of f is defined and normal is called the *Fatou set* of f and its complement $J(f) = \overline{\mathbb{C}} \setminus F(f)$ is called the *Julia set* of f.

A set S is called *completely invariant* with respect to the meromorphic function f if $f^n(S) \subset S$ for all $n \in \mathbb{Z}$, where $f^{-k}(z) = \{w : f^k(w) = z\}$ for $k \in \mathbb{N}$

Now, we introduce the notion of periodic points [5, 6, 7, 15] which play an important role in the iteration theory.

A point $z_0 \in \overline{\mathbb{C}}$ is called a *periodic point* of f if $f^n(z_0) = z_0$ for some $n \geq 1$. In this case, the smallest n with this property is called the *period* of z_0. A point z_0 is called *preperiodic* if $f^t(z_0)$ is periodic for some integer $t \geq 1$. In particular, a periodic point of period 1 is called a *fixed point.* For a periodic point z_0 of period n, $(f^n)'(z_0)$ is called the *multiplier*, or *eigenvalue* of z_0. If $z_0 = \infty$, the multiplier is defined to be $(g^n)'(0)$, where g is the conjugation of f with the inversion map $z \mapsto \frac{1}{z}$.

A periodic point z_0 is called *attracting, indifferent, or repelling* according to the modulus of its multiplier is less than, equal to, or greater than 1. A periodic point of multiplier 0 is called *super-attracting.* The multiplier of an indifferent periodic point is of the form $e^{2\pi i\alpha}$, where $0 \leq \alpha < 1$. We say that z_0 is *rationally indifferent* if α is rational and *irrationally indifferent* otherwise. If z_0 is a periodic point of period n, then $\alpha = \{z_0, f(z_0), \cdots, f^{n-1}(z_0)\}$ is called a *cycle* of length n.

The zeros of f' and the multiple poles of f are called *critical points* of f and the images of critical points are called *critical values.* We denote by C_f and by C_f^+ the set of all critical points of f and the forward images of C_f, respectively.

A point $w \in \overline{\mathbb{C}}$ is called an *asymptotic value* for f if there exists a path $\alpha : [0, 1) \to \mathbb{C}$ such that $\lim_{t\to 1} \alpha(t) = \infty$ and $\lim_{t\to 1}(f \circ \alpha)(t) = w$. The path α is called a *critical path* for w. We denote by $\text{sing}(f^{-1})$ the set of all critical and finite asymptotic values of f and (finite) limit points of these values.

A function f is said to be a *critically finite function* or *CF-function* if the set $\text{sing}(f^{-1})$ is finite.

There is a considerable relationship between the Julia set and periodic points of a meromorphic function.

THEOREM 2.1. [6, Theorem 3] *If f is a meromorphic function, then $J(f)$ is perfect.*

THEOREM 2.2. [3, Theorem 1] *If f is a meromorphic function, then $J(f)$ is the closure of the set of repelling periodic points of f.*

Let U be a component of the Fatou set $F(f)$ of a function f. Then U is called *periodic* if $f^m(U) = U$ for some $m \in \mathbb{N}$. In this case, the class $C = \{U, f(U), \cdots, f^{m-1}(U)\}$ is called the *periodic cycle.* If $f^n(U) \neq f^m(U)$ for any $n \neq m$, then U is called a *wandering domain.*

R. Goldberg and L. Keen [13] proved the following theorem.

THEOREM 2.3. *A CF-entire transcendental function has no wandering domains.*

A similar result was obtained in [11] as follows;

THEOREM 2.4. *Let f be an entire transcendental function. If there exists a finite set $\{a_1, \cdots, a_q\}$ of points such that*

$$f : \overline{\mathbb{C}} \setminus f^{-1}\{a_1, \cdots, a_q\} \to \overline{\mathbb{C}} \setminus \{a_1, \cdots, a_q\}$$

is a covering map, then f has no wandering domains.

The following classification of periodic components of a Fatou set was initiated by Baker, Kotus, and Lü [4]. In the case for rational functions, it seems to have been given by Sullivan [16].

THEOREM 2.5. [4, Theorem 2.2] *Let f be a meromorphic function and U a periodic component of $F(f)$ of period p. Then we have one of the following possibilities ;*

(1) *U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, and U is called the immediate attractive basin of z_0.*

(2) *∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$, we have $(g^p)'(0) = 1$ where $g(z) = 1/f(1/z)$.) In this case, U is called a Leau domain.*

(3) *There exists an analytic homeomorphism $\phi : U \to D$, where D is the unit disk such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Siegel disc.*

(4) *There exists an analytic homeomorphism $\phi : U \to A$, where $A = \{z : 1 < |z| < r\}, r > 1$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Herman ring.*

(5) *There exists* $z_0 \in \partial U$ *such that* $f^{np}(z) \to z_0$ *for* $z \in U$ *as* $n \to \infty$, *but* $f^p(z_0)$ *is not defined. In this case,* U *is called a Baker domain.*

Note that the case of an immediate attractive basin is sometimes further distinguished on whether the attracting periodic point contained in it is superattracting or not. If this is the case, then U is called a *Böttcher domain*, otherwise U is called a *Schröder domain.*

REMARK 2.6. Eremenko and Lyubich [11] showed that a CF-entire transcendental function does not have Baker domains. Also this result was generalized to CF-meromorphic functions [6].

3. Completely invariant components

For a (CF-)meromorphic function f, the following properties which were shown in [4] and [6] are used in the proof of Theorem 3.9.

LEMMA 3.1. [4, Lemma 4.1] *Let* U *be a completely invariant component of* $F(f)$. *Then the number of components of* ∂U *is 1 or* ∞.

LEMMA 3.2. [4, Lemma 4.2] *If* U *is a completely invariant component of* $F(f)$, *then* $\partial U = J(f)$.

LEMMA 3.3. [4, Lemma 4.3] *If there are two or more completely invariant components of* $F(f)$, *then each is simply connected.*

THEOREM 3.4. [4, Theorem 4.5] *If* f *is a CF-meromorphic function, then* $F(f)$ *has at most two completely invariant components.*

THEOREM 3.5. [6, Theorem 12] *A CF-meromorphic function has no wandering domains.*

THEOREM 3.6. [6, Theorem 7] *Let* f *be a meromorphic function and let* $C = \{U_0, U_1, \cdots, U_{p-1}\}$ *be a periodic cycle of components of* $F(f)$.

(1) *If C is a cycle of immediately attractive basins or Leau domains, then $U_j \cap sing(f^{-1}) \neq \emptyset$ for some $j \in \{0, 1, \cdots, p-1\}$. More precisely, there exists $j \in \{0, 1, \cdots, p-1\}$ such that $U_j \cap sing(f^{-1})$ contains a point which is not preperiodic or such that U_j contains a periodic critical point (in which case C is a cycle of Böttcher domains).*

(2) *If C is a cycle of Siegel discs or Herman rings, then ∂U_j is contained in $\overline{\cup_{n \geq 0}\{f^n(z) : z \in sing(f^{-1})\}}$ for all $j \in \{0, 1, \cdots, p-1\}$.*

REMARK 3.7. From the previous theorem, the number of cycles of immediate attractive basins and Leau domains does not exceed the number of singularities of f^{-1}.

A boundary point w of a domain U is said to be an *accessible point* of ∂U if there exists an arc $\alpha : [0, 1) \to U$ such that $\lim_{t \to 1} \alpha(t) = w$.

LEMMA 3.8. [4, Lemma 4.4] *Suppose that f is a CF-transcendental meromorphic function, and that $F(f)$ has a simply connected completely invariant component U. Then ∞ is an accessible point of ∂U.*

We end up this section with the proof of the following theorem.

THEOREM 3.9. *Let f be a CF-meromorphic function. Suppose that U_1 and U_2 are two completely invariant components of $F(f)$. Then $F(f) = U_1 \cup U_2$.*

Proof. Assume that $W = F(f) \setminus (U_1 \cup U_2) \neq \emptyset$. For each $j = 1, 2$, $\partial U_j = J(f)$ by Lemma 3.2 and ∞ is an accessible point of ∂U_j by Lemma 3.8. So there are two arcs $\gamma_j \subset U_j$ such that $\gamma_j \to \infty$. Let p be a pole of f. Then $\gamma_j' = f^{-1}(\gamma_j) \to p$ and since U_j are completely invariant, $\gamma_j' \subset U_j$. So p is an accessible point of ∂U_j. Also by Lemma 3.3, each U_j is simply connected. Thus we can find two cross cuts Γ_j of U_j such that each of which has one end at p and the other at ∞.

Then $\Gamma = \Gamma_1 \cup \Gamma_2$ forms a Jordan curve in $\overline{\mathbb{C}}$ which separates $\overline{\mathbb{C}} \setminus \Gamma$ into two components D and E.

Note that W is not connected by Theorem 3.4. Let V be a component of W. Since $F(f)$ is open, we have that $\partial V \subset J(f)$.

Now take two points $\zeta_j \in \Gamma_j$ where ζ_j are not p and ∞. Then we can join ζ_1 and ζ_2 by an arc δ which passes a point $w \in \partial V$ in D (or E). So we can choose a point $\xi \in \delta$ such that $B_\epsilon(\xi) \subset V$ for sufficiently small $\epsilon > 0$, and then $B_\epsilon(\xi)$ meets U_1 or U_2. But it is impossible. □

4. Dynamics of Entire Transcendental Functions

Whether the Julia set $J(e^z)$ is the whole plane or not was a long standing open question which was conjectured by Fatou [12] in 1926. It was first proved by Misiurewicz [14] in 1981, surprisingly simple argument without using any sophisticated theorems.

THEOREM 4.1. [14] *The Julia set $J(E)$ of $E(z) = e^z$ is the whole plane.*

Many authors studied about the Julia sets of entire transcendental functions, notably, Fatou [12] showed that the set of periodic points is dense in the Julia set $J(f)$ of an entire function f, and Baker, Kotus and Lü [3] extended this result to show that $J(f)$ is the closure of the set of repelling periodic points. Furthermore, the Julia set of a *CF*-entire transcendental function is the closure of the set of points whose orbits tend to infinity. Also R. Devaney showed that the Julia set of $E_\lambda(z) = \lambda\exp(z)$, $0 < \lambda < \frac{1}{e}$, contains a Cantor bouquet by constructing a homeomorphism [8].

In this section, we give another proof that $J(E_\lambda)$, $0 < \lambda < \frac{1}{e}$, contains a Cantor bouquet by employing the Devaney and Tangerman's argument.

For a quadratic polynomial, there are basically two types of Julia sets : one is a Cantor set and the other is a connected set. For a CF-entire transcendental function, there is a similar dichotonomy : either the Julia set is the entire plane or it contains a Cantor bouquet [6].

Now we consider the definition of the Cantor bouquet [8].

For $N \in \mathbb{N}$, let $\Sigma_N = \{\underline{s} = (s_0, s_1, \cdots) \mid s_j \in \mathbb{Z}, |s_j| \leq N\}$ be the set of all sequences of integers between $-N$ and N. Then, with the product topology, Σ_N is homeomorphic to a Cantor set. The shift map $\sigma : \Sigma_N \to \Sigma_N$ is defined by $\sigma(s_0, s_1, s_2, \cdots) = (s_1, s_2, s_3, \cdots)$.

For a CF-entire transcendental function E, we call an invariant subset $C_N \subset J(E)$ a *Cantor N-bouquet* if there exists a homeomorphism $h : \Sigma_N \times [1, \infty) \to C_N$ such that

(1) $\pi \circ h^{-1} \circ E \circ h(\underline{s}, t) = \sigma(\underline{s})$, where $\pi : \Sigma_N \times [1, \infty) \to \Sigma_N$ is the natural projection map,

(2) $\lim_{t \to \infty} h(\underline{s}, t) = \infty$,

(3) $\lim_{n \to \infty} (E^n \circ h)(\underline{s}, t) = \infty$ if $t \neq 1$.

A Cantor N-bouquet is similar to a Cantor set, but the components are curves tending to infinity instead of points. The invariance of C_N requires that $E(h(\underline{s}, 1)) = h(\sigma(\underline{s}), 1)$. Hence the set Λ of points $h(\underline{s}, 1)$ is an invariant set on which E is topologically conjugate to the shift map. We call Λ the *crown* of C_N. The curve $h(\underline{s}, t)$ for $t > 1$ is called the *tail associated to* $\underline{s}$.

Let C_N be a Cantor N-bouquet and suppose that $C_N \subset C_{N+1} \subset \cdots$ is an increasing sequence of Cantor N-bouquets. Then the set

$$C = \overline{\bigcup_{N \geq 0} C_N}$$

is called a *Cantor bouquet*.

Now we recall the specific conditions guaranteeing that the set of points whose orbits remain inside a given exponential tract T, defined below, is a Cantor bouquet.

Let E be a CF-entire transcendental function and D be an open disk in the plane which contains all of the critical and (finite) asymptotic values. Put $\Gamma = D^c$.

THEOREM 4.2. [10, Theorem 1.1] (1) *Any connected component T of $E^{-1}(\Gamma)$ is a disk whose closure contains ∞.*

(2) $E : T \to \Gamma$ *is a universal covering map.*

We call a component T of $E^{-1}(\Gamma)$ an *exponential tract.*

Proof. We claim that $E : E^{-1}(\Gamma) \to \Gamma$ is a covering map. Let $z \in \text{int}(\Gamma)$ and let B be an open disk about z in Γ. For $w \in E^{-1}(z)$ and $b \in B$, choose a simple path γ in B connecting z and b. Since z is not a critical value, we can define a local inverse E_w^{-1} for E on a neighborhood of z which is uniquely determined by the condition $E_w^{-1}(z) = w$. Since B does not contain singular values, we can analytically continue E_w^{-1} along γ to b.

Define a map

$$\psi : E^{-1}(z) \times B \to E^{-1}(B)$$

by $\psi(w, b) = E_w^{-1}(b)$. By the Monodromy theorem, ψ is well defined, and also it is one to one and onto since it is the inverse function. So ψ is a homeomorphism and $E|_{E^{-1}(\Gamma)}$ is a covering map.

Since Γ is a punctured disk in $\overline{\mathbb{C}}$, any component T is either a punctured disk or a disk. If T is punctured at ∞, then E is a polynomial, and if it is punctured at $a \neq \infty$, then E has a pole at a. But these are impossible since E is entire transcendental. Thus T is a disk, and hence $E|_T$ is a universal covering map. $\square$

Now we assume that there is a disk B_ρ which is disjoint from the exponential tract T such that $E : T \to B_\rho^c$ is a covering map and that T is contained in a sector S. Then we may fix a ray $\zeta(r) = re^{i\theta_0}$ which is disjoint from S and defined for $r \geq \rho$.

Let $\gamma_j(r)$, $j \in \mathbb{Z}$, denote the preimages of $\zeta(r)$ in T. We may choose the index j in the natural way so that γ_j and γ_{j+1} are adjacent for each j. Then $\gamma_j(r) \to \infty$ as $r \to \infty$ and the curves γ_j and γ_{j+1} bound a strip which serve as a fundamental domain for $E|_T$. Let T_j be the fundamental domain bounded by γ_j and γ_{j+1} and let $W_N = \cup_{j=-N}^{N} T_j$.

We say that $E|_T$ has an *asymptotic direction* θ^* if $\gamma_j(r)$ is C^1-asymptotic to a straight line with direction θ^* for each curve γ_j defining the fundamental domains. So if $E|_T$ has an asymptotic direction θ^*, then each of the curves γ_j which lies in W_N meets a circle of sufficiently large radius at the unique point.

An exponential tract T is said to be *hyperbolic* if there exist positive constants R_1, α, and C such that

(1) $|E(z)| > C \cdot e^{r^\alpha}$

(2) $|E'(z)| > C \cdot e^{r^\alpha}$

(3) $|\arg(E'(z))| < C \cdot e^{-r^\alpha}$

for all z in W_N such that $E(z)$ lies in W_N and $|z| = r \geq R_1$.

We will prove Theorem 4.5 by employing the following theorem and lemma.

THEOREM 4.3. [10, Theorem 3.3] *Let E be a CF-entire transcendental function. Let T be a hyperbolic exponential tract on which E has an asymptotic direction θ^*. Then for each N*

$$\Lambda_N = \{z \in W_N | E^j(z) \in W_N \text{ for all } j \geq 0\}$$

is a Cantor N-bouquet. Consequently,

$$J_T(E) = \{z : E^j(z) \in T \text{ for all } j \geq 0\}$$

contains a Cantor bouquet.

LEMMA 4.4. [10, Lemma 2.3] *If both z and $E_\lambda(z)$ lie in W_N, then $Re(E_\lambda(z)) > 2Re(z)$.*

THEOREM 4.5. *The Julia set $J(E_\lambda)$ of $E_\lambda(z) = \lambda e^z$, $0 < \lambda < 1/e$, contains a Cantor bouquet.*

Proof. Clearly 0 is the only singular value for E_λ. Let D be the unit disk and let $\Gamma = D^c$. Then $0 \in D$ and

$$E_\lambda^{-1}(\Gamma) = \{z : \mathrm{Re}(z) > \ln(1/\lambda)\}$$

is the only exponential tract T and is contained in the sector $S = \{z : |\arg z| < \pi/2\}$.

Fix a ray $\zeta(t) = te^{i\pi}$, $t \geq 1$. Then it is disjoint from S and $E_\lambda^{-1}(\zeta)$ consists of curves γ_j for all $j \in \mathbb{Z}$, where $\gamma_j = \{\eta + (2j-1)\pi i : \eta \geq \ln(1/\lambda)\}$. Clearly, each γ_j is C^1-asymptotic to $\theta^* = 0$. Then we denote by T_j the fundamental domain bounded by the curves γ_j and γ_{j+1}, that is,

$$T_j = \{z : \mathrm{Re}(z) > \ln(1/\lambda), (2j-1)\pi < \mathrm{Im}(z) < (2j+1)\pi\} .$$

Now we claim that T is a hyperbolic exponential tract. Take a positive real number R_1 larger than $(2N+1)\pi$ and let $r \geq R_1$. Then for $z \in W_N$ so that $E_\lambda(z) \in W_N$ and $|z| = r$, we have

$$|E_\lambda(z)| = \lambda e^{\mathrm{Re}(z)} > \lambda e^{\sqrt{r^2-(2N+1)^2\pi^2}} .$$

Also since $\mathrm{Re}((E_\lambda(z))) > 2\mathrm{Re}(z)$ by Lemma 4.4, we have

$$\begin{aligned} |\arg(E'(z))| &< \arctan \frac{(2N+1)\pi}{\mathrm{Re}(E(z))} \\ &< \arctan \frac{(2N+1)\pi}{2\sqrt{r^2-(2N+1)^2\pi^2}} . \end{aligned}$$

Since the arctangent is bounded, we may choose the positive real numbers C and α which satisfy the following inequalities (1) and (2). Indeed, α is taken so that r^α is sufficiently close to 1.

$$\lambda e^{\sqrt{r^2-(2N+1)^2\pi^2}} > C \cdot e^{r^\alpha} \tag{1}$$

$$C > e^{r^\alpha} \cdot \arctan \frac{(2N+1)\pi}{2\sqrt{r^2-(2N+1)^2\pi^2}}\ . \tag{2}$$

So T is a hyperbolic exponential tract which has an asymptotic direction $\theta^* = 0$. Thus by Theorem 4.3,

$$\Lambda_N = \{z \in W_N | E_\lambda^j(z) \in W_N \text{ for all } j \geq 0\}$$

is a Cantor N-bouquet and

$$J_T(E_\lambda) = \{z : E_\lambda^j(z) \in T \text{ for all } j \geq 0\}$$

contains a Cantor bouquet. □

References

[1] L. Ahlfors, *Complex Analysis*, McGraw-Hill International Edition, 1979.

[2] I. N. Baker, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **11** (1975), 277–283.

[3] I. N. Baker, J. Kotus and Y. Lü, *Iterates of meromorphic functions I*, Ergodic Theory Dynamical Systems **11** (1991), 241–248.

[4] ———, *Iterates of meromorphic functions III: Periodic Domains*, Ergodic Theory Dynamical Systems **11** (1991), 603–618.

[5] A. F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.

[6] W. Bergweiler, *Iteration of meromorphic functions*, Bull. of A.M.S. **29** (1993), 151–188.

[7] R. Devaney, *An Introduction to Chaotic Dynamical Systems* 2nd ed., Addison-Wesely Publishing Comp. Ins., 1989.

[8] ———, *Complex Dynamics and Entire Functions*, Proc. of Symposia in Appl. Math. **49** (1994), 181–206.

[9] R. Devaney and M. Krych, *Dynamics of exp(z)*, Ergodic Theory Dynamical Systems **4** (1984), 35–52.

[10] R. Devaney and F. Tangerman, *Dynamics of entire functions near the essential singularity*, Ergodic Theory Dynamical Systems **6** (1986), 489–503.

[11] A. E. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020.

[12] P. Fatou, *Sur l'itération des fonctions transcendentes entiéres*, Acta Math. **47** (1926), 337–360.

[13] R. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynamical Systems **6** (1986), 183–192.

[14] M. Misiurewicz, *On iterates of* e^z, Ergodic Theory Dynamical Systems **1** (1981), 103–106.

[15] N. Steinmetz, *Rational Iteration*, Walter de Gruyter, Berlin, 1993.

[16] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418.

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